PERIODIC HOMEOMORPHISMS ON $S^2 \times (0,1)$ AND R^3

BY

DAVID G. WINSLOW

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1979

ACKNOWLEDGEMENTS

I would like to express my appreciation to the members of my supervisory committee, and in particular to Gerhard Ritter, for suggesting this problem to me. His enthusiasm for mathematics was largely responsible for my interest in the area of topology. I would also like to thank Dave Wilson for his helpful suggestions.

To my father, I would like to express my gratitude for his support and his many sacrifices, both large and small, that have allowed me to pursue a career in mathematics.

Above all, I would like to thank my wife Alicia for her continued encouragement and moral support. It is to her that this is dedicated.

TABLE OF CONTENTS

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

PERIODIC HOMEOMORPHISMS ON $S^2 \times (0,1)$ and R^3

By

David G. Winslow

August 1979

Chairman: Gerhard X. Ritter
Major Department: Mathematics

It has been conjectured that all periodic homeomorphisms acting piecewise linearly on a given 3-manifold must be conjugate to the standard periodic homeomorphisms on that manifold. Many results have been obtained when the given 3-manifold is compact. For example, it has been shown that if $h: F \times I \to F \times I$ is an involution, where F denotes any surface and I denotes the unit interval, then h is topologically equivalent to a product involution. Recently, analogous results have been obtained for various noncompact 3-manifolds. The results obtained in the noncompact case have, for the most part, been restricted to PL involutions on certain 3-manifolds. In this dissertation, it is shown that $S^2 \times (0,1)$ admits exactly four nonequivalent cyclic 2^n actions, $n \geq 2$, and R^3 admits exactly two nonequivalent cyclic 2^n actions, $n \geq 2$, up to weak conjugation, where S^2

and R^3 denote the standard 2-sphere and Euclidean 3-space, respectively. The method used in each case is to first classify all Z_4 actions and then extend the classification to arbitrary 2^n actions, $n \geq 2$, primarily using the Lifting Theorem. It is also established that $S^2 \times (0,1)$ admits no free Z_m actions for $m > 2$. K. W. Kwun and J. L. Tollefson have shown that $S^2 \times (0,1)$ admits exactly seven nonequivalent PL involutions and R^3 admits exactly three nonequivalent PL involutions up to conjugation. Their results together with the results obtained in this dissertation provide a complete classification of all cyclic 2^n actions, $n \geq 1$, on $S^2 \times (0,1)$ and R^3 up to weak conjugation.

INTRODUCTION

It has been conjectured that all periodic homeomorphisms acting piecewise linearly on a given 3-manifold must be conjugate to the standard periodic homeomorphisms on that manifold. Much work has been done on this problem when the given 3-manifold is compact (see [8], [9], [16], and [20]). Recently, analogous results have been obtained by Ritter and Clark [17], and Kwun and Tollefson [11], for various noncompact 3-manifolds. The results obtained in the noncompact case have, for the most part, been restricted to PL involutions on certain 3-manifolds. In this dissertation, we show that $S^2 \times (0,1)$ admits exactly four nonequivalent cyclic 2^n actions, $n \geq 2$, and R^3 admits exactly two nonequivalent cyclic 2^n actions, $n \geq 2$, up to weak conjugation.

Chapter One contains a brief review of PL topology relating to 3-manifolds as well as basic notation and definitions of terms which will be used in the sequel. The next two chapters are devoted to the classification of Z_4 actions on $S^2 \times (0,1)$. Suppose that $h: S^2 \times (0,1) \to S^2 \times (0,1)$ is a semifree Z_4 action on $S^2 \times (0,1)$. In Chapter Two, we construct a 2-sphere, S, in $S^2 \times (0,1)$ such that S does not bound a 3-cell and either S is h-invariant or $S \cap hS = \emptyset$ with $h^2S = S$. This 2-sphere

plays a central role in the classification of Z_4 actions on both $S^2 \times (0,1)$ and R^3. Since S does not bound a 3-cell, S separates $S^2 \times (0,1)$ into two components, A and B, each homeomorphic to $S^2 \times (0,1)$. In the event that S is h-invariant, either both A and B are h-invariant or $hA = B$. If A and B are h-invariant, we obtain a decomposition of $S^2 \times (0,1)$ into $S^2 \times (0,1) = (\cup_{n=1}^{\infty} C_n) \cup (\cup_{n=-1}^{-\infty} C_n)$, where each C_n is h-invariant and homeomorphic to $S^2 \times [0,1]$. If $S \cap hS = \emptyset$ or $hS = S$ and $hA = B$, $S^2 \times (0,1)$ may be decomposed into $S^2 \times (0,1) = D_0 \cup (\cup_{n=1}^{\infty} D_n \cup hD_n)$, where D_0 is a 3-annulus invariant under h, and each D_n is an h^2-invariant 3-annulus. In Chapter Three, we first classify all Z_4 actions on $S^2 \times [0,1]$. Thus, if

$$S^2 \times (0,1) = (\cup_{n=1}^{\infty} C_n) \cup (\cup_{n=-1}^{-\infty} C_n),$$

there is a homeomorphism $\alpha_n\colon C_n \to S^2 \times [0,1]$, for each n, such that $\alpha_n h \alpha_n^{-1}(x,t) = (s(x),t)$, where $s\colon S^2 \to S^2$ is a standard action on S^2 and $(x,t) \in S^2 \times (0,1)$. Furthermore, by a slight alteration of the sequence of homeomorphisms, $\{\alpha_n\}_{n=-\infty}^{\infty}$, a homeomorphism $\alpha\colon S^2 \times (0,1) \to S^2 \times (0,1)$ is constructed so that $\alpha h \alpha^{-1}(x,t) = (s(x),t)$. A similar technique is used in the case that

$$S^2 \times (0,1) = D_0 \cup (\cup_{n=1}^{\infty} D_n \cup hD_n).$$

The nonfree Z_4 actions are handled in a somewhat easier manner.

In Chapter Four, we obtain a classification of all cyclic 2^n actions on $S^2 \times (0,1)$, $n \geq 2$. It is also shown that $S^2 \times (0,1)$ admits no free Z_m actions for $m>2$. It follows from Theorem 1 [11] and Theorem A [10] that there are seven nonequivalent PL involutions on $S^2 \times (0,1)$, and thus all cyclic 2^n actions on $S^2 \times (0,1)$ are classified up to weak conjugation, $n \geq 1$.

The remainder of Chapter Four is devoted to the classification of 2^n actions on R^3, $n \geq 2$. We first show that if $h: R^3 \to R^3$ is a Z_4 action then fix(h) is either a single point or an open arc. If fix(h) is homeomorphic to an open arc, then fix(h^k) is homeomorphic to an open arc for $1 \leq k < 4$. That h is a standard follows from Theorem 1 [12] and a simple application of the lifting theorem. If fix(h) is homeomorphic to a point, then $R^3 - N$ is homeomorphic to $S^2 \times (0,1)$ and is invariant under h, where N is a regular neighborhood of fix(h). The techniques used for $S^2 \times (0,1)$ are then applied to construct a conjugation map $\alpha: Cl(R^3 - N) \to Cl(R^3 - B)$, where B denotes the standard unit 3-ball, which can easily be extended to N by coning over the boundary of N to obtain a homeomorphism $\alpha: R^3 \to R^3$ such that $\alpha h \alpha^{-1} = s$, where s is a standard action. We extend the classification to arbitrary 2^n actions, $n \geq 2$, primarily using the lifting theorem. It follows from Theorem 1 [12], and Theorem 1 [11], that there are three nonequivalent involutions on R^3 and thus all cyclic 2^n actions on R^3 are classified, $n \geq 1$.

CHAPTER ONE
PRELIMINARIES

The first section of this chapter is devoted to a brief review of PL topology relating to 3-manifolds. The definitions and development found in [5] and [7] are followed closely. We define only those concepts which will be needed in the sequel, and we refer the reader to [7] for a more general treatment. The second section includes general notation, definitions, and two preliminary theorems appearing in other sources which will be used in the remaining chapters. Specific terms which have limited use will be defined in later chapters as the need arises.

PL Topology of 3-Manifolds

Let R^n denote Euclidean n-space. If σ is a simplex in R^n, for some n, and $\{a_0, a_1, \ldots, a_k\}$ are its vertices, then the notation $\langle a_0, a_1, \ldots, a_k \rangle$ will be used to represent σ if there is a need to specify its vertices. We will view a <u>simplicial complex</u> as a locally finite collection, K, of simplexes in R^n, for some n, satisfying:

i. If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.

ii. If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is a face of both σ and τ.

The underlying space of K will be denoted by $|K| = \cup \{\sigma | \sigma \in K\}$. A simplicial complex L is a <u>subdivision</u>

of K if $|K| = |L|$ and each simplex of L lies in some simplex of K. For simplicial complexes K_1 and K_2, a map $f: |K_1| \to |K_2|$ is <u>piecewise linear</u>, or a PL map, provided there are subdivisions L_1 of K_1 and L_2 of K_2 with respect to which f is simplicial, i.e., f takes vertices of L_1 to vertices of L_2 and takes each simplex of L_1 linearly, in terms of barycentric coordinates, onto a simplex of L_2.

Now, let K be a simplicial complex considered as a subset of R^n for some n. If τ and σ are simplexes in K, we use the notation, $\tau<\sigma$, to indicate τ is a face of σ. The <u>star of σ in K</u>, denoted by $st(\sigma,K)$ is defined by $st(\sigma,K) = \{\tau \in K | \sigma<\tau\}$. The <u>link of σ in k</u>, denoted by $lk(\sigma,K)$, is defined as $lk(\sigma,K) = \{\tau \in K | \tau \cap \sigma = \emptyset$ and there is a $\Delta \in K$ such that $\tau<\Delta$ and $\sigma<\Delta\}$. Suppose $\rho = \langle a_0,a_1,\ldots,a_\ell\rangle$ and $\gamma = \langle b_0,b_1,\ldots,b_m\rangle$ are simplexes in R^n. We say that ρ and γ are <u>joinable in R^n</u> if $\rho \cap \gamma = \emptyset$ and the set $\{a_0,a_1,\ldots,a_\ell,b_0,b_1,\ldots,b_m\}$ is in general position in R^n. If ρ and γ are joinable in R^n, then we define the <u>join</u> of ρ with γ, $\rho*\gamma$, by

$$\rho*\gamma = \langle a_0,a_1,\ldots,a_\ell,b_0,b_1,\ldots,b_m\rangle.$$

If ρ and γ are joinable in R^n and $\rho*\gamma \in K$, then we say that ρ and γ are <u>joinable in K</u>. Two complexes M and N are joinable in R^n if

i. $\rho*\gamma$ exists for all $\rho \in M$, $\gamma \in N$.

ii. $(\rho_1*\gamma_1) \cap (\rho_2*\gamma_2) < \rho_1*\gamma_1$ and
$(\rho_1*\gamma_1) \cap (\rho_2*\gamma_2) < \rho_2*\gamma_2$

for all $\rho_1,\rho_2 \in M$ and $\gamma_1,\gamma_2 \in N$.

If M and N are joinable, their join is denoted by $M*N$ and defined by $M*N = M \cup N \cup \{\rho*\gamma | \rho \in M \text{ and } \gamma \in N\}$. If $\sigma \in K$, then we may view K as $K = (\sigma*L) \cup R$, where $L = lk(\sigma,K)$ and $R = \{\eta \in K \mid \eta \not\leq \sigma*L\}$. Now let $\sigma' = \{\tau \in K | \tau$ is a proper face of $\sigma\}$, and let $a \in int(\sigma)$, where $int(\sigma)$ denotes the interior of σ. The complex $K' = (a*\sigma'*L) \cup R$ is called the complex derived from K by starring K at $a \in int(\sigma)$. A subdivision $K^{(1)}$ of a complex K is called a first derived subdivision of K if it is obtained by starring each simplex σ of K, in order of increasing dimension, at exactly one interior point. An n^{th} derived subdivision, $K^{(n)}$, is defined inductively to be $K^{(n)} = (K^{(n-1)})^{(1)}$. The barycentric first derived subdivision of K is the first derived subdivision of K obtained by starring at the barycenters of the simplexes, and the n^{th} derived barycentric subdivision is defined analogously.

A topological n-manifold is a separable metric space each of whose points have an open neighborhood homeomorphic to R^n or $R^n_+ = \{(x_1,x_2,\ldots,x_n) \in R^n | x_n \geq 0\}$. The interior of M, denoted $int(M)$, is the set of all points of M having neighborhoods homeomorphic to R^n, and the boundary of M, denoted by ∂M, is $M-int(M)$. A manifold is closed if it is compact and has empty boundary, and it is open if it has no compact component and has empty boundary.

A triangulation of an n-manifold M is a pair, (T,s), where T is a simplicial complex and $s\colon |T| \to M$ is a homeomorphism. Two triangulations (T_1,s_1) and (T_2,s_2) are

compatible provided $s_2^{-1}s_1\colon |T_1| \to |T_2|$ is piecewise linear. A triangulation (T,s) of an n-manifold, M, is combinatorial provided that for each vertex v of T, $|lk(v,T)|$ is piecewise linearly homeomorphic to an (n-1)-simplex or the boundary of an n-simplex. A PL structure on a manifold M is a maximal, nonempty collection of compatible combinatorial triangulations of M. By a PL manifold we will mean a manifold, M, together with a PL structure on M. It follows from the triangulation theorem due to Bing [2] and Moise [14] that every 3-manifold has a unique PL structure up to PL homeomorphism. A submanifold, N, of a PL manifold, M, is a PL submanifold if there is a triangulation (T,s) in the PL structure on M and a subcomplex L of T such that $(L,s|_L)$ is a combinatorial triangulation of N.

Now let M be a 3-manifold and $h\colon M \to M$ a periodic homeomorphism. We say that h is a PL homeomorphism if there is a PL structure on M and triangulations (T_1,s_1) and (T_2,s_2) within the PL structure such that $s_2^{-1}hs_1\colon |T_1| \to |T_2|$ is piecewise linear. Since $s_1^{-1}s_2\colon |T_2| \to |T_1|$ is piecewise linear, we may assume that $s_1^{-1}hs_1\colon |T_1| \to |T_1|$ is simplicial for a suitable subdivision of T_1.

Notation, Definitions, and Preliminary Theorems

Unless otherwise stated, we will assume that all spaces, subspaces, and maps are in the PL category; S^n and R^n will denote the n-sphere and Euclidean n-space, respectively. An n-cell is any space PL homeomorphic to an n-simplex.

An <u>n-annulus</u> is any space homeomorphic to $S^{n-1} \times [0,1], n \geq 2$. An <u>open n-annulus</u> is simply the interior of an n-annulus. The closure, boundary, and interior of a set, A, will be denoted by Cl(A), ∂A, and int(A), respectively.

Given a homeomorphism $h: M \to M$ of a period n, where M is a 3-manifold, $\text{fix}(h^k) = \{x \in M | h^k(x) = x\}$ will denote the fixed point set of h^k, $1 \leq k < n$. We say that h is a <u>semifree</u> action if $\text{fix}(h) = \emptyset$, and $\text{fix}(h^k) \neq \emptyset$ for <u>some</u> k, $1 < k < n$; h is a <u>nonfree</u> action if $\text{fix}(h^k) \neq \emptyset$ for each k, $1 \leq k < n$; and h is a <u>free</u> action if $\text{fix}(h^k) = \emptyset$ for all k, $1 \leq k < n$. The symbol $\langle h^k \rangle$ will denote the group of homeomorphisms generated by h^k, $1 \leq k \leq n$. If $s: M \to M$ is also a homeomorphism, we say that h is <u>conjugate</u> to s provided there is a homeomorphism $\alpha: M \to M$ such that $\alpha h \alpha^{-1} = s$. If the homeomorphism α satisfies the weaker condition that $\langle \alpha h \alpha^{-1} \rangle = \langle s \rangle$, we say that h is <u>weakly conjugate</u> to s. Finally, a homeomorphism $g: \text{int}(M) \to \text{int}(M)$ can be <u>extended</u> to M if there is a homeomorphism $g': M \to M$ such that $g'|_{\text{int}(M)}$ is conjugate g.

The following two theorems are used numerous times in the following chapters and are listed here for easy reference.

Theorem 1.1 (Theorem 1 [11]). Let M be a compact 3-manifold and h and involution of int(M). Then h can be extended to M. Furthermore, the extension of h is unique up to conjugation.

Theorem 1.2 (Theorem A [10]). Let F be a compact surface and let h be a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$, where I denotes the unit interval. Then there exists a map g of F (with g^2 = identity) such that h is equivalent to the involution h' of $F \times I$ defined by $h'(x,t) = (g(x), \lambda(t)$ for $(x,t) \in F \times I$ and $\lambda(t) = t$ or $\lambda(t) = 1-t$.

CHAPTER TWO
A SPECIAL 2-SPHERE IN $S^2 \times (0,1)$

Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a semifree Z_4 action on $S^2 \times (0,1)$. In this chapter, we construct a 2-sphere, S, in $S^2 \times (0,1)$ such that S does not bound a 3-cell and either $hS = S$ or $S \cap hS = \emptyset$ with the property that $h^2S = S$. This 2-sphere will play a central role in both the classification of Z_4 actions on $S^2 \times (0,1)$ and R^3. In the first section, it is shown that h^2 is conjugate to the standard involution $s: S^2 \times (0,1) \to S^2 \times (0,1)$ defined by $s(x,t) = (\sigma(x),t)$, where $\sigma: S^2 \to S^2$ is the standard period 2 rotation of S^2 leaving the poles fixed. Thus, there is a 2-sphere, $\overline{S}$, such that $\overline{S}$ does not bound a 3-cell and $h^2\overline{S} = \overline{S}$. Next, $\overline{S}$ is altered to a 2-sphere, S', that satisfies the following properties:

i. S' does not bound a 3-cell in $S^2 \times (0,1)$;

ii. $h^2S' = S'$;

iii. $S' \cap hS' = \emptyset$ or $S' \cap hS'$ consists of a finite number of simple closed curves at which S' and hS' meet <u>transversely</u>. To clarify condition iii, we say that S' and hS' meet transversely at a simple closed curve $J \subseteq S' \cap hS'$ if $N(J,S') \cap A_i \neq \emptyset$, $i = 1,2$, where A_1 and A_2 are the components of $S^2 \times (0,1) - hS'$ and $N(J,S')$ is a regular

neighborhood of J in S'. Finally, if $S' \cap hS' \neq \emptyset$, S' is altered to a 2-sphere, S, satisfying the desired properties.

The Involution h^2

If $h: S^2 \times (0,1) \to S^2 \times (0,1)$ is a Z_4 action, then $h^2: S^2 \times (0,1) \to S^2 \times (0,1)$ is an involution. The following theorem classifies h^2 up to conjugation.

Theorem 2.1. Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a Z_4 action. Then there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\alpha h^2 \alpha^{-1}(x,t) = (\sigma(x),t)$, $x \in S^2$ and $t \in (0,1)$, where $\sigma: S^2 \to S^2$ is the standard period 2 rotation on S^2 leaving the poles fixed.

Proof of Theorem 2.1.

By Theorem 1.1, there is a PL involution $h': S^2 \times [0,1] \to S^2 \times [0,1]$ and a homeomorphism $\beta: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\beta h^2 \beta^{-1} = h'|_{S^2 \times (0,1)}$. By Theorem 1.2, there is a homeomorphism $\rho': S^2 \times [0,1] \to S^2 \times [0,1]$ such that $\rho' h' \rho'^{-1}(x,t) = (s(x),\lambda(t))$, where $s: S^2 \to S^2$ is an involution and $\lambda(t) = t$ or $\lambda(t) = 1-t$. Let

$$\rho = \rho'|_{S^2 \times (0,1)} \quad \text{and} \quad \alpha = \rho\beta.$$

Then $\alpha h^2 \alpha^{-1}(x,t) = (s(x),\lambda(t))$.

Let $D = [S^2 \times (0,1) \cup \{-\infty,\infty\}]$ denote the two point compactification of $S^2 \times (0,1)$. Then D is homeomorphic to S^3, and there is a homeomorphism (not necessarily PL) $h^*: D \to D$ which extends h. Clearly h^{*2} is orientation preserving, and $\mathrm{fix}(h^{*2}) \neq \emptyset$ since $\{-\infty,\infty\} \subseteq \mathrm{fix}(h^{*2})$. By Smith's Theorem [18], $\mathrm{fix}(h^{*2})$ is homeomorphic to S^1, and hence $\mathrm{fix}(h^2) = [\mathrm{fix}(h^{*2}) - \{-\infty,\infty\}]$ is homeomorphic to two open arcs.

It now follows easily that $\alpha h^2 \alpha^{-1}(x,t) = (\sigma(x),t)$, where $\sigma: S^2 \to S^2$ is the standard period 2 rotation on S^2 leaving the poles fixed.

Theorem 2.1, aside from classifying the action h^2, provides some interesting intrinsic properties which we shall use later. First, there is a 2-sphere, S, which does not bound a 3-cell such that $h^2 S = S$. Secondly, $\mathrm{fix}(h^2)$ is homeomorphic to two open arcs, and any 2-sphere $S \subseteq S^2 \times (0,1)$ which does not bound a 3-cell has nonempty intersection with $\mathrm{fix}(h^2)$. Finally, we have the following important corollary.

Corollary. $S^2 \times (0,1)$ admits no free Z_4 actions.

A General Positioning Lemma

Results similar to the following general positioning lemma have appeared numerous times in the literature, usually without proof. For example, it is known that if h is a periodic homeomorphism acting freely on a 3-manifold,

M, and $T \subseteq \mathrm{int}(M)$ is a surface invariant under a subgroup $\langle h^k \rangle$ of $\langle h \rangle$, then there is a surface T' isotopic to T which is also invariant under the subgroup $\langle h^k \rangle$ and either $T' \cap hT' = \emptyset$ or $T' \cap hT'$ consists of a finite number of simple closed curves at which T' and hT' meet transversely (see [15]). We obtain a similar result under the weaker condition that h acts semifreely.

Lemma 2.2. If $h: S^2 \times (0,1) \to S^2 \times (0,1)$ is a semi-free Z_4 action, then there is a 2-sphere, S, in $S^2 \times (0,1)$ satisfying the following properties:

i. S does not bound a 3-cell;

ii. $h^2 S = S$;

iii. $S \cap hS = \emptyset$ or $S \cap hS$ consists of a finite number of simple closed curves at which S and hS meet transversely.

Proof of Lemma 2.2.

By Theorem 2.1, there is a 2-sphere, S, satisfying conditions i and ii, with $S \cap \mathrm{fix}(h^2)$ consisting of two points. Suppose S fails to satisfy condition iii. S will be altered by arbitrarily small isotopies to satisfy condition iii, while preserving properties i and ii. We will assume the following without loss of generality:

1. $h: S^2 \times (0,1) \to S^2 \times (0,1)$ is simplicial for a triangulation, (M,t), of $S^2 \times (0,1)$;

2. the simplexes, σ, of M satisfy the property that $th^k t^{-1}\sigma \cap \sigma = \emptyset$ whenever $\sigma \cap t^{-1}(\mathrm{fix}(h^2)) = \emptyset$, $1 \le k < 4$.

From property 1 above we have that $t^{-1}ht\colon M \to M$ is simplicial. To simplify notation let $\bar{h} = t^{-1}ht$ and $\bar{S} = t^{-1}S$. Note that $\bar{S} \cap \bar{h}\bar{S}$ is a subcomplex of M. We remove "bad" simplexes from $\bar{S} \cap \bar{h}\bar{S}$ in order of decreasing dimension so that the resulting 2-sphere, $t\bar{S}'$, will satisfy the conclusion of the lemma. We first alter $\bar{S}$ to a 2-sphere, $\bar{S}'$, which has the properties that $\bar{S}' \cap \bar{h}\bar{S}'$ contains no 2-simplexes, $h^2(t\bar{S}') = t\bar{S}'$, and $t\bar{S}'$ does not bound a 3-cell.

1. Suppose $|\Delta^2| \subseteq |\bar{S} \cap \bar{h}\bar{S}|$, where Δ^2 is a 2-simplex in M. Then $\mathrm{int}(\Delta^2) \cap \mathrm{fix}(h^2) = \emptyset$ since we assume that $\mathrm{fix}(h^2)$ is a subcomplex of M.

Case a. Suppose $\Delta^2 \cap \mathrm{fix}(\bar{h}^2) = \emptyset$. Now Δ^2 has exactly two 3-simplexes in M which have Δ^2 as a face, i.e., $\mathrm{st}(\Delta^2, M) = A \cup B$, where A and B are 3-simplexes. We star A at $a \in \mathrm{int}(A)$ and similarly star $\bar{h}^k A$ at $\bar{h}^k a \in \mathrm{int}(\bar{h}^k A)$. Then $a * \Delta^2$ is a 3-simplex containing Δ^2 as a face. Now let $\sigma = a * \partial\Delta^2$ and

$$\bar{S}' = [(\bar{S} - (\Delta^2 \cup \bar{h}^2\Delta^2)) \cup (\sigma \cup \bar{h}^2\sigma)].$$

Clearly $\bar{h}^2\bar{S}' = \bar{S}'$, $\bar{S}'$ does not bound a 3-cell, and $\bar{S} \cap \bar{h}\bar{S}$ consists of fewer 2-simplexes. We denote the new subdivision by M again and let $\bar{S}' = \bar{S}$ for simplicity.

Case b. Suppose $\Delta^2 \cap \mathrm{fix}(\bar{h}^2) \neq \emptyset$. Then $\Delta^2 \cap \mathrm{fix}(h^2)$ is a point since $S \cap \mathrm{fix}(h^2)$ consists of two points and $\mathrm{fix}(\bar{h}^2)$ is a subcomplex of M. If $\Delta^2 = \langle a_1, a_2, a_3 \rangle$, where $a_1 \in \mathrm{fix}(\bar{h}^2)$, then $\bar{h}^k(\langle a_2, a_3 \rangle) \cap \langle a_2, a_3 \rangle = \emptyset$, $1 \le k < 4$. Hence, $\Delta^2 \cap \bar{h}^2\Delta^2 = a_1$ and $\bar{h}\Delta^2 \cap \bar{h}^3\Delta^2 = \bar{h}a_1$, where

$\overline{h}a_1 \in \text{fix}(\overline{h}^2)$. Again, Δ^2 has two 3-simplexes in M, say A and B, having Δ^2 as a face. We follow the same procedure as in case a to eliminate Δ^2 from $\overline{S} \cap \overline{h}\,\overline{S}$. Since $\Delta^2 \neq \overline{h}^k\Delta^2$, $1 \leq k \leq 4$, $A \neq \overline{h}^2 A$ and it follows that $\text{int}(A) \cap \text{int}(\overline{h}^2 A) = \emptyset$. Let

$$\overline{S}' = [(\overline{S} - (\Delta^2 \cup \overline{h}^2\Delta^2)) \cup (\sigma \cup \overline{h}^2\sigma)],$$

where σ is defined as in case a, and copy the subdivision under the interates of $\overline{h}$. Clearly $\overline{S}'$ does not bound a 3-cell, $\overline{h}^2\overline{S} = \overline{S}$, and $\overline{S}' \cap \overline{h}\,\overline{S}'$ has fewer 2-simplexes in its intersection. We denote the new subdivision by M again and let $\overline{S}' = \overline{S}$.

Using the above procedure a finite number of times, we can construct a 2-sphere, $S' = t\overline{S}'$, such that $h^2 S' = S'$, S' does not bound a 3-cell, and $\overline{S}' \cap \overline{h}\,\overline{S}'$ is a complex of some subdivision, M', of M under which h is simplicial. For simplicity, let $M' = M$, $S' = S$, and $\overline{S}' = \overline{S}$.

2. We now remove 1-simplexes from $\overline{S} \cap \overline{h}\,\overline{S}$ satisfying the following property:

(*) If $|\Delta^1| \subseteq |\overline{S} \cap \overline{h}\,\overline{S}|$ and $A = \text{st}(\Delta^1, \overline{h}\,\overline{S})$, then $A \cap E_1 = \emptyset$ or $A \cap E_2 = \emptyset$ where E_1 and E_2 are the components of $M - \overline{S}$, i.e., $\overline{S}$ and $\overline{h}\,\overline{S}$ do not meet transversely at Δ^1.

Case a. Suppose $|\Delta^1| \subseteq |\overline{S} \cap \overline{h}\,\overline{S}|$, Δ^1 satisfies property (*), and $\Delta^1 \cap \text{fix}(\overline{h}^2) = \emptyset$. We assume that M is sufficiently subdivided so that $\text{st}(\Delta^1, M) \cap \text{st}(\overline{h}\Delta^1, M) = \emptyset$, $1 \leq k < 4$. Let $M^{(2)}$ be a second derived subdivision of M under which $\overline{h}$ is simplicial.

Let $D^* = st(\Delta^1, M)$. The 2-sphere $\overline{S}$ will be altered within D^* and corresponding copies of the alteration will be made under the iterates of $\overline{h}$. In $M^{(2)}$,

$$|\Delta^1| = |<a_0,a_1>| \cup |<a_1,a_2>| \cup |<a_2,a_3>| \cup |<a_3,a_4>|,$$

where a_0 and a_4 are vertices in M, a_2 is a vertex in $M^{(1)}$, and a_1 and a_3 are vertices in $M^{(2)}$. Let

$$D = [st(<a_0,a_1>,M^{(2)}) \cup st(a_2,M^{(2)}) \cup st(<a_3,a_4>,M^{(2)})].$$

Now each subcomplex in this union is a 3-ball, and it can be shown that $|st(<a_0,a_1>,M^{(2)}) \cap st(a_2,M^{(2)})|$ and $|st(<a_3,a_4>,M^{(2)}) \cap st(a_2,M^{(2)})|$ are disks, and $st(<a_0,a_1>,M^{(2)}) \cap st(<a_3,a_4> \cap M^{(2)}) = \emptyset$. Hence, D is a 3-ball and $B = |D \cap \overline{S}^{(2)}|$ is a spanning disk for D. Also, ∂B separates ∂D into two components whose closures are disks, say B_1 and B_2. It follows easily from property (*) that either $B_1 \cap \overline{h}\,\overline{S} = \{a_0,a_4\}$ or $B_2 \cap \overline{h}\,\overline{S} = \{a_0,a_4\}$. Without loss of generality, assume that $B_1 \cap \overline{h}\,\overline{S} = \{a_0,a_4\}$. Let

$$\overline{S}' = [(\overline{S} - (B \cup \overline{h}^2 B)) \cup (B_1 \cup \overline{h}^2 B_1)].$$

Clearly $\overline{S}'$ does not bound a 3-cell, $\overline{h}^2\overline{S}' = \overline{S}'$, and by construction $\overline{S}' \cap \overline{h}\,\overline{S}'$ has fewer 1-simplexes of type (*) with no new intersections involving 2-simplexes introduced. We let $\overline{S}' = \overline{S}$, $M^{(2)} = M$ and continue.

Case b. Suppose $|\Delta^1| \subseteq |\overline{S} \cap \overline{h}\,\overline{S}|$, Δ^1 satisfies property (*), and $\Delta^1 \cap \operatorname{fix}(\overline{h}^2) \neq \emptyset$. Since $\overline{S} \cap \operatorname{fix}(\overline{h}^2)$

consists of two points and $\text{fix}(\bar{h}^2)$ is a subcomplex of M it follows that $\Delta^1 \cap \text{fix}(\bar{h}^2)$ consists of a single point. We again let $D^* = \text{st}(\Delta^1, M)$, and assume that M is sufficiently subdivided so that $D^* \cap \bar{h}\, D^* = \emptyset$. We perform the same modifications as in case a to produce

$$\bar{S}' = [(\bar{S} - (B \cup \bar{h}^2 B)) \cup (B_1 \cup \bar{h}^2 B_1)],$$

where B and B_1 are defined as in case a. By construction, $\bar{S}' \cap \bar{h}\,\bar{S}'$ has fewer 1-simplexes satisfying property (*), and no 2-simplexes are introduced in the intersection. To verify that $\bar{S}'$ is a nonsingular 2-sphere, it suffices to show that $\text{int}(\text{st}(\Delta^1, M)) \cap \text{int}(\text{st}(\bar{h}^2\Delta^1, M)) = \emptyset$. We first note that $|\text{int}(\text{st}(\Delta^1, M))| = \{x \in |M| \mid x \in \text{int}(\tau^3)$ and $\Delta^1 < \tau^3$; or $x \in \text{int}(\tau^2)$ and $\Delta^1 < \tau^2$; or $x \in \text{int}(\Delta^1)\}$, and similarly $|\text{int}(\text{st}(\bar{h}^2\Delta^1, M))| = \{x \in |M| \mid x \in \text{int}(\tau^3)$ and $\bar{h}^2\Delta^1 < \tau^3$; or $x \in \text{int}(\tau^2)$ and $\bar{h}^2\Delta^1 < \tau^2$; or $x \in \text{int}(\bar{h}^2\Delta^1)\}$, where the τ^i are i-simplexes in $M, i = 2,3$.

We claim that $\text{int}(\text{st}(\Delta^1, M)) \cap \text{int}(\text{st}(\bar{h}^2\Delta^1, M)) = \emptyset$. Suppose there is a 3-simplex $|\tau^3| \subseteq |\text{st}(\Delta^1, M) \cap \text{st}(\bar{h}^2\Delta^1, M)|$. Then $\Delta^1 < \tau^3$ and $\bar{h}^2\Delta^1 < \tau^3$. If $\Delta^1 = \langle a_0, a_1 \rangle$, where $a_0 \in \text{fix}(\bar{h}^2)$, then $\bar{h}^2\Delta^1 = \langle a_0, \bar{h}^2 a_1 \rangle$ with $\bar{h}^2 a_1 \neq a_1$. Thus $\langle a_1, \bar{h}^2 a_1 \rangle < \tau^3$, and it follows that $\bar{h}^2(\langle a_1, \bar{h}^2 a_1 \rangle) = \langle a_1, \bar{h}^2 a_1 \rangle$. In this case, $\text{fix}(\bar{h}^2) \cap \text{int}(\langle a_1, \bar{h}^2 a_1 \rangle) \neq \emptyset$, contradicting the assumption that $\text{fix}(\bar{h}^2)$ is a subcomplex of M. Hence, there are no 3-simplexes in $\text{st}(\Delta^1, M) \cap \text{st}(\bar{h}^2\Delta^1, M)$. Next suppose there is a 2-simplex $|\tau^2| \subseteq |\text{st}(\Delta^1, M) \cap \text{st}(\bar{h}^2\Delta^1, M)|$. Then $\Delta^1 < \tau^2$ and

$\bar{h}^2\Delta^1 < \tau^2$, and it follows that $\tau^2 = \langle a_0, a_1, \bar{h}^2 a_1\rangle$. By an argument similar to the one above, we obtain the contradiction that $\langle a_1, \bar{h}^2 a_1\rangle \cap \mathrm{fix}(\bar{h}^2) \neq \emptyset$. The verification that $\mathrm{int}(\mathrm{st}(\Delta^1, M)) \cap \mathrm{int}(\mathrm{st}(\bar{h}^2\Delta^1, M)) = \emptyset$ is now trivial, and the claim is established.

Thus, $\bar{S}'$ is a nonsingular 2-sphere, and clearly $\bar{S}'$ does not bound a 3-cell. Again we let $\bar{S}' = \bar{S}$ and $M^{(2)} = M$ for simplicity.

By repeating the above alterations a finite number of times, we can construct a 2-sphere, $S' = t\bar{S}'$, such that S' does not bound a 3-cell, $h^2 S' = S'$ and $\bar{S}' \cap \bar{h}\bar{S}'$ has only vertices in $M^{(2)}$ as intersection points and 1-simplexes at which $\bar{S}'$ and $\bar{h}\bar{S}'$ meet transversely. For simplicity, let $S' = S$, $\bar{S}' = \bar{S}$, and $M^{(2)} = M$.

3. We now remove vertices, Δ^0, from $\bar{S} \cap \bar{h}\bar{S}$ with the following property:

$$(**) \quad \mathrm{st}(\Delta^0, \bar{h}\bar{S}) \cap E_1 = \emptyset \text{ or } \mathrm{st}(\Delta^0, \bar{h}\bar{S}) \cap E_2 = \emptyset,$$

where E_1 and E_2 are the components of $M-\bar{S}$.
Again let $M^{(2)}$ be a second derived subdivision of M under which h is simplicial, and let $|\Delta^0| \subseteq |\bar{S} \cap \bar{h}\bar{S}|$ satisfy property (**).

Case a. Suppose $|\Delta^0| \not\subseteq \mathrm{fix}(\bar{h}^2)$. Then $\mathrm{st}(\Delta^0, M^{(2)}) \cap \mathrm{st}(\bar{h}^k\Delta^0, M^{(2)}) = \emptyset$, $1 \leq k < 4$. Also $B = \mathrm{st}(\Delta^0, \bar{S}^{(2)})$ is a spanning disk for $\mathrm{st}(\Delta^0, M^{(2)})$, and ∂B separates $\partial(\mathrm{st}(\Delta^0, M^{(2)})$ into two components whose closures are disks, say B_1 and B_2. Because of property (**), either $\mathrm{int}(B_1) \cap \bar{h}\bar{S} = \emptyset$ or $\mathrm{int}(B_2) \cap \bar{h}\bar{S} = \emptyset$. Without loss of generality,

assume that $\operatorname{int}(B_1) \cap \overline{h}\,\overline{S} = \emptyset$ and let

$$\overline{S}' = [(\overline{S} - (B \cup \overline{h}^2 B)) \cup (B_1 \cup \overline{h}^2 B_1)].$$

Clearly $\overline{h}^2\overline{S}' = \overline{S}'$, $\overline{S}'$ does not bound a 3-cell, and it is easy to verify that $\overline{S} \cap \overline{h}\,\overline{S}$ has fewer vertices of type (**) without creating any new intersections. Let $\overline{S}' = \overline{S}$, and $M^{(2)} = M$ for simplicity.

Case b. Suppose $|\Delta^0| \subseteq \operatorname{fix}(\overline{h}^2)$, and assume that M is sufficiently subdivided so that $\operatorname{st}(\Delta^0, M) \cap \operatorname{st}(\overline{h}\Delta^0, M) = \emptyset$. Since $|\Delta^0| \subseteq \operatorname{fix}(\overline{h}^2)$, $\overline{h}^2(\operatorname{st}(\Delta^0, M^{(2)}) = \operatorname{st}(\Delta^0, M^{(2)})$. Moreover, there is a 1-simplex, σ^1, such that $|\sigma^1| \subseteq \operatorname{st}(\Delta^0, M^{(2)})$ and $|\sigma^1| \subseteq \operatorname{fix}(\overline{h}^2)$. Otherwise, Δ^0 is an isolated point of $\operatorname{fix}(h^2)$, contradicting the fact that $\operatorname{fix}(h^2)$ is homeomorphic to two open arcs. Now, $\operatorname{st}(\Delta^0, \overline{S}^{(2)})$ is a spanning disk for $D = \operatorname{st}(\Delta^0, M^{(2)})$, and $B = \partial(\operatorname{st}(\Delta^0, \overline{S}^{(2)}))$ separates ∂D into two components whose closures, B_1 and B_2, are disks. It follows from property (**) that either $\operatorname{int}(B_1) \cap \overline{h}\,\overline{S} = \emptyset$ or $\operatorname{int}(B_2) \cap \overline{h}\,\overline{S} = \emptyset$. Assume $\operatorname{int}(B_1) \cap \overline{h}\,\overline{S} = \emptyset$. Now ∂B and ∂D are invariant under h^2, and hence $\overline{h}^2 B_1 = B_1$ or $\overline{h}^2 B_1 = B_2$. But $\sigma^1 \cap \operatorname{int}(B_i) \neq \emptyset$ for $i = 1$ or $i = 2$, where $\sigma^1 \subseteq \operatorname{fix}(\overline{h}^2)$. Thus, $\overline{h}^2 B_i = B_i$, $i = 1,2$ and we let $\overline{S}' = [(\overline{S} - B) \cup B_1]$. Then $\overline{S}'$ does not bound a 3-cell, $\overline{h}^2\overline{S}' = \overline{S}'$, and $\overline{S}' \cap \overline{h}\,\overline{S}'$ has fewer vertices satisfying property (**) with no new intersections introduced. For simplicity we let $\overline{S}' = \overline{S}$, $M^{(2)} = M$ and continue.

By repeating the above procedure a finite number of times, we obtain a 2-sphere, $S' = t\overline{S}'$, such that S' does

not bound a 3-cell, $h^2S' = S'$, and $\overline{S}' \cap \overline{h}\,\overline{S}'$ consists of 1-simplexes where $\overline{S}'$ and $\overline{h}\,\overline{S}'$ meet transversely. Again let $S' = S$, $\overline{S}' = \overline{S}$, and $M^{(2)} = M$, for simplicity. At this stage it can be shown that the components of $S \cap hS$ consist of closed curves.

4. We finally remove vertices from $\overline{S} \cap \overline{h}\,\overline{S}$ that have the following property:

(***) $st(\Delta^0, \overline{S} \cap \overline{h}\,\overline{S})$ consists of more than two 1-simplexes. These are the saddle points.

Case a. Suppose $|\Delta^0| \not\subseteq fix(\overline{h}^2)$. Again we may assume that M is sufficiently subdivided so that $st(\Delta^0,M) \cap \overline{h}^k(st(\Delta^0,M)) = \emptyset$, $1 \le k < 4$. Let $D = st(\Delta^0, M^{(2)})$, where $M^{(2)}$ is a second derived subdivision under which h is simplicial. Now $B = D \cap \overline{S}$ is a spanning disk for D, and ∂B separates ∂D into two components whose closures are disks, B_1 and B_2. We let

$$\overline{S}' = [(\overline{S} - (B \cup \overline{h}^2 B)) \cup (B_1 \cup \overline{h}^2 B_1) \ .$$

Then $\overline{S}'$ does not bound a 3-cell, $\overline{h}^2\overline{S}' = \overline{S}'$, no new intersections of type 1, 2, or 3 are introduced, and each new vertex, v, of $[(\overline{S}' \cap \overline{h}\,\overline{S}') \cap B_1]$ has the property that $st(v, \overline{S}' \cap \overline{h}\,\overline{S}')$ consists of two 1-simplexes.

Case b. Suppose $|\Delta^0| \subseteq fix(\overline{h}^2)$. Assume that M is sufficiently subdivided so that $st(\Delta^0,M) \cap \overline{h}(st(\Delta^0,M)) = \emptyset$. Let $D = st(\Delta^0, M^{(2)})$. Then $B = D \cap \overline{S}^{(2)}$ is a spanning disk for D, and ∂B separates ∂D into two components whose closures are disks, B_1 and B_2. As in case b

of type 3 intersections, it follows that $\bar{h}^{2}B_1 = B_1$. Let $\bar{S}' = [(\bar{S} - B) \cup B_1]$. Then $\bar{h}^2\bar{S}' = \bar{S}'$, $\bar{S}'$ does not bound a 3-cell, no new intersections of type 1, 2, or 3 are introduced, and each new vertex, v, of $[(\bar{S}' \cap \bar{h}\bar{S}') \cap B_1]$ has the property that $st(v, \bar{S}' \cap \bar{h}\bar{S}')$ consists of two 1-simplexes. Let $\bar{S}' = \bar{S}$, $M^{(2)} = M$, and continue.

Eliminating the remaining vertices with property (***), we obtain a 2-sphere, $S = t\bar{S}$, satisfying the following:

i. S does not bound a 3-cell;

ii. $h^2S = S$;

iii. $S \cap hS = \emptyset$ or $S \cap hS$ consists of a finite number of simple closed curves at which S and hS meet transversely. Hence, Lemma 2.2 is established.

It should be noted that in the proof of Lemma 2.2 each alteration was performed within a regular neighborhood of a simplex and only involved replacing a disk with another disk to improve the intersection of S with hS. This lemma therefore has the following useful generalization. Let M be a 3-manifold, and $g\colon M \to M$ a semifree Z_4 action. Let T denote the fixed point set of g^2, and furthermore suppose T is a 1-manifold. Also let C be a surface in the interior of M. Define T and C to be in general position if for every point $p \in T \cap C$, there is a regular neighborhood, N, of p such that $(N \cap T) \cap C = p$ and the components of $N - (N \cap C)$, N_1 and N_2, satisfy the property that $N_i \cap T \neq \emptyset$, $i = 1,2$.

In other words, T is locally piercing with respect to C at p. Using the same methods as those used in the proof of Lemma 2.2, we obtain the following generalization:

Theorem 2.3. Let M be a 3-manifold admitting a semi-free Z_4 action, $g: M \to M$, with $fix(g^2)$ homeomorphic to a 1-manifold. Furthermore, suppose there is a surface C in $int(M)$ such that $g^2C = C$ and C and $fix(g^2)$ are in general position. Then there is a surface C' satisfying the following properties:

i. C' is isotopic to C.

ii. $g^2C' = C'$.

iii. $C' \cap gC' = \emptyset$ or $C' \cap gC'$ consists of a finite number of simple closed curves at which C' and hC' meet transversely.

A Special 2-Sphere in $S^2 \times (0,1)$

In this section, we construct a 2-sphere, S, with the properties that S does not bound a 3-cell and either $hS = S$ or $S \cap hS = \emptyset$ with $h^2S = S$. The following result will be needed in the construction.

Lemma 2.4. Suppose E_1, E_2, and E_3 are three disks in $S^2 \times (0,1)$ having only their boundaries in common. If the 2-spheres $E_1 \cup E_2$ and $E_1 \cup E_3$ do not bound 3-cells, then $E_2 \cup E_3$ must bound a 3-cell.

Proof of Lemma 2.4.

By taking the two point compactification of $S^2 \times (0,1)$, we may view $S^2 \times (0,1)$ as a subset of $S^3 = S^2 \times (0,1) \cup \{-\infty,\infty\}$. Since $E_1 \cup E_2 \cup E_3$ is compact, there are numbers t_0, $t_1 \in (0,1)$, $t_1 > t_0$, such that $E_1 \cup E_2 \cup E_3 \subseteq S^2 \times [t_0,t_1]$. Hence, by coning over $S^2 \times t_0$ and $S^2 \times t_1$, we may assume that E_1, E_2, and E_3 are polyhedral subsets of S^3. By Alexander's Theorem [1], $E_1 \cup E_2$ separates S^3 into two components whose closures, B_1 and B_2, are 3-balls with $\partial B_1 = \partial B_2 = E_1 \cup E_2$. Without loss of generality, assume that $E_3 \subseteq B_1$. Now $B_i \cap \{-\infty,\infty\} \neq \emptyset$, $i = 1,2$, since $E_1 \cup E_2$ does not bound a 3-cell in $S^2 \times (0,1)$. Moreover, E_3 is a spanning disk for B_1, i.e., $E_3 \subseteq B_1$ and $\partial B_1 \cap E_3 = \partial E_3$, and E_3 separates B_1 into two components whose closures, C_1 and C_2, are 3-balls with $\partial C_1 = E_1 \cup E_3$ and $\partial C_2 = E_2 \cup E_3$. If $\{-\infty,\infty\} \cap \text{int}C_2 \neq \emptyset$, then $\{-\infty,\infty\} \cap \text{int}(C_1) = \emptyset$ and $E_1 \cup E_3$ bounds a 3-cell in $S^2 \times (0,1)$, a contradiction. Hence $\{-\infty,\infty\} \cap \text{int}C_2 = \emptyset$ and $E_2 \cup E_3$ bounds a 3-cell, namely C_2 in $S^2 \times (0,1)$.

We are now ready to construct the 2-sphere announced at the beginning of this chapter.

Lemma 2.5. Given a semifree Z_4 action $h: S^2 \times (0,1) \to S^2 \times (0,1)$ there is a 2-sphere, S, that does not bound a 3-cell such that $hS = S$ or $S \cap hS = \emptyset$ with the property that $h^2S = S$.

Proof of Lemma 2.5.

Let S be the 2-sphere in Lemma 2.2. If $S \cap hS = \emptyset$, then S is the desired 2-sphere. Therefore, suppose that $S \cap hS \neq \emptyset$. At each stage of the construction we alter S to a 2-sphere, S', such that S' does not bound a 3-cell, and either $hS' = S'$ or $S' \cap hS'$ is a strict subset of $S \cap hS$, with $h^2S' = S'$. First, let D be an innermost disk on hS, i.e., $\mathrm{int}(D) \cap S = \emptyset$, with boundary J, where $J \subseteq S \cap hS$. J divides S into two disks, D_1 and D_2. By Theorem 2.1, both S and hS have the property that $S \cap \mathrm{fix}(h^2)$ consists of two points and $hS \cap \mathrm{fix}(h^2)$ consists of two points. Without loss of generality, assume that $hD \subseteq D_1$. It follows easily that either $D \cup D_1$ does not bound a 3-cell or $D \cup D_2$ does not bound a 3-cell (see [19]). We have two major cases to consider.

Case 1. Suppose $D \cup D_1$ does not bound a 3-cell. Now either $hJ = J$ or $J \cap hJ = \emptyset$.

a. If $hJ = J$, then $hD = D_1$ and either $h^2D = D$ or $h^2D = \mathrm{Cl}(hS - D)$. But if $h^2D = \mathrm{Cl}(hS-D)$, $D_1 \cup \mathrm{Cl}(hS - D)$ does not bound a 3-cell since $h(D \cup D_1) = D_1 \cup \mathrm{Cl}(hS - D)$. This is a contradiction by making the following observation. Consider the three disks D, D_1, and $\mathrm{Cl}(hS - D)$. Since $D \cup D_1$ and $hS = D \cup \mathrm{Cl}(hS - D)$ do not bound 3-cells, it follows from Lemma 2.4 that $D_1 \cup \mathrm{Cl}(hS - D)$ must bound a 3-cell. Hence, $h^2D = D$. If we let $S' = D \cup D_1$, then $hS' = S'$ and S' satisfies the conclusion of the lemma.

b. Suppose $J \cap hJ = \emptyset$.

i. If $\mathrm{int}(D) \cap \mathrm{fix}(h^2) \neq \emptyset$, $h^2D = D$ since D is innermost, and by an argument similar to that in case a, it can be shown that $h^2D_1 = D_1$. Now, choose a simple closed curve $J' \subseteq D_1$ sufficiently close to J such that the annulus, A, bounded by J and J' has the properties that $A \cap hS = J$ and $h^2A = A$. Let D' be a disk sufficiently close to D such that $D' \cap hD' = D' \cap hS = \emptyset$, $h^2D' = D'$, $D' \cap S = \partial D' = J'$, and $S' = (D_1 - A) \cup D'$ does not bound a 3-cell. By construction, $S' \cap hS' = (D_1 - A) \cup h(D_1 - A)$ is a strict subset of $S \cap hS$ and $h^2S' = S'$.

ii. If $\mathrm{fix}(h^2) \cap \mathrm{int}(D) = \emptyset$, then we claim that $h^2D_2 \subseteq D_1$. Now $h^2J \neq J$, for otherwise $h^2D = \mathrm{Cl}(hS - D)$ is innermost, contrary to our assumption that $hJ \neq J$. Since $h^2J \neq J$, it follows that $J \cap \mathrm{fix}(h^2) = \emptyset$. Thus, either $h^2J \subseteq \mathrm{int}(D_1)$ or $h^2J \subseteq \mathrm{int}(D_2)$. If $h^2D_2 \subseteq \mathrm{int}(D_1)$, then $h^2D_2 \subseteq D_1$ since $h^2J \neq J$. Now suppose $h^2J \subseteq \mathrm{int}(D_2)$. Then either $h^2D_2 \subseteq D_2$ or $D_2 \cup h^2D_2 = S$. But $h^2D_2 \subseteq D_2$ implies that $h^2D_2 = D_2$ and $h^2J = J$, a contradiction. If $D_2 \cup h^2D_2 = S$ and $D_2 \cap \mathrm{fix}(h^2) = \emptyset$, then $S \cap \mathrm{fix}(h^2) = \emptyset$, again a contradiction. Hence, to establish the claim it only remains to prove that $D_2 \cap \mathrm{fix}(h^2) = \emptyset$. Suppose $D_2 \cap \mathrm{fix}(h^2) \neq \emptyset$. Again either $h^2J \subseteq \mathrm{int}(D_1)$ or $h^2J \subseteq \mathrm{int}(D_2)$. If $h^2J \subseteq \mathrm{int}(D_1)$, then $D_2 \subseteq h^2D_2$ since $D_2 \cap h^2D_2 \neq \emptyset$. It also follows that $h^2D_2 \subseteq h^2(h^2D_2) = D_2$. Therefore, $h^2D_2 = D_2$ and $h^2J = J$ contradicting the fact that $h^2J \neq J$. If $h^2J \subseteq \mathrm{int}(D_2)$, either $h^2D_2 \subseteq D_2$ or

$D_2 \cap h^2D_2 = A$, where A is an annulus with the property that $\mathrm{fix}(h^2) \cap S \subseteq A$. If $h^2D_2 \subseteq D_2$, it follows easily that $h^2D_2 = D_2$ and $h^2J = J$, a contradiction. If $D_2 \cap h^2D_2 = A$, then $D \cup D_1$ is a nontrivial 2-sphere with $(D \cup D_1) \cap \mathrm{fix}(h^2) = \emptyset$, contradicting the fact that every 2-sphere which does not bound a 3-cell has nonempty intersection with $\mathrm{fix}(h^2)$. Thus, $D_2 \cap \mathrm{fix}(h^2) = \emptyset$ and we have the claim.

We now alter S to remove J as an intersection curve. First let $S^* = \mathrm{Cl}(S - (D_2 \cup h^2D_2)) \cup (D \cup h^2D)$. S^* does not bound a 3-cell since $D \cup D_2$ bounds a 3-cell and S is nontrivial. Because $(D \cup h^2D) \cap (hD \cup h^3D) \subseteq (D \cup h^2D) \cap S$ and D and h^2D are innermost, we have introduced no new intersections in $S^* \cap hS^*$. S^* will now be altered to remove J as an intersection curve. Let $J \subseteq D_1$ be a simple closed curve sufficiently close to J so that the annulus, A, bounded by J and J' has the property that $A \cap hS = J$. Now, choose a disk, D', sufficiently close to D such that $D' \cap hD' = D' \cap hS = \emptyset$, $D' \cap S = J'$, and $D' \cap h^2D' = \emptyset$. Let

$$S' = [(D' \cup h^2D') \cup (D_1 - (A \cup h^2A \cup h^2D_2))].$$

To verify that $h^2S' = S'$, it suffices to show that

$$h^2(\mathrm{Cl}(D_1 - (A \cup h^2A \cup h^2D_2))) = \mathrm{Cl}(D_1 - (A \cup h^2A \cup h^2D_2)).$$

However, $[(D_2 \cup A) \cup (h^2D_2 \cup h^2A)]$ is clearly invariant under h^2 and hence, $[S - (D_2 \cup A \cup h^2D_2 \cup h^2A)] = [D_1 - (A \cup h^2A \cup h^2D_2)]$ is h^2-invariant also. By construction

$S' \cap hS'$ is a strict subset of $S \cap hS$, and by choosing S' sufficiently close to S^*, S' will not bound a 3-cell.

Case 2. Suppose that $D \cup D_2$ does not bound a 3-cell. Again we assume that $hD \subseteq D_1$. Now either $hJ = J$ or $J \cap hJ = \emptyset$.

a. If $J \cap hJ = \emptyset$, then $J \cap \text{fix}(h^2) = \emptyset$. Otherwise, $J \cap \text{fix}(h^2)$ consists of a single point since $\text{fix}(h^2)$ is invariant under h. But if $J \cap \text{fix}(h^2)$ is a single point, $h^2J = J$, and since $S \cap \text{fix}(h^2)$ consists of two points, there is an $a \in \text{fix}(h^2)$ such that $a \in \text{int}(D_1)$ or $a \in \text{int}(D_2)$. Thus, we have that $h^2|_{D_1}$ is an involution on D_1 such that $\text{fix}(h^2|_{D_1})$ contains one or two points, one being on the boundary of D_1, a contradiction. Hence, $J \cap \text{fix}(h^2) = \emptyset$.

i. If $h^2J = J$, then since $\text{fix}(h^2) \cap J = \emptyset$, $D \cap \text{fix}(h^2) \neq \emptyset$ and $h^2D = D$. By the same argument it follows that $h^2D_2 = D_2$. In this case, we let $S' = D \cup D_2$. Then $h^2S' = S'$, and $S' \cap hS' = D_2 \cap hD_2$ is a strict subset of $S \cap hS$.

ii. If $h^2J \neq J$, we claim that $h^2D_1 \subseteq D_2$. Since D is innermost, $h^2J \subseteq D$ and therefore $h^2J \subseteq hS - D$. If $D \cap \text{fix}(h^2) \neq \emptyset$, it follows that $D \subseteq h^2D$ and $J \subseteq \text{int}(h^2D)$ which contradicts the fact that D is innermost. Therefore, $D \cap \text{fix}(h^2) = \emptyset$. Now suppose $D_1 \cap \text{fix}(h^2) = \emptyset$. Either $h^2J \subseteq \text{int}(D_1)$ or $h^2J \subseteq \text{int}(D_2)$. If $h^2J \subseteq \text{int}(D_1)$, then $h^2D_1 \subseteq D_1$ or $D_1 \cap h^2D_1 = A$, where $S \cap \text{fix}(h^2) \subseteq A$. But $h^2D_1 \subseteq D_1$ implies that $h^2D_1 = D_1$ and $h^2J = J$, contrary to our assumption that $h^2J \neq J$. If $h^2D_1 \cap D_1 = A$, then

$D \cup D_2$ is a nontrivial 2-sphere with $(D \cup D_2) \cap \text{fix}(h^2) = \emptyset$, contradicting the fact that every 2-sphere which does not bound a 3-cell in $S^2 \times (0,1)$ has nonempty intersection with $\text{fix}(h^2)$. Thus, $D_1 \cap \text{fix}(h^2) = \emptyset$. Again either $h^2J \subseteq D_1$ or $h^2J \subseteq D_2$. If $h^2J \subseteq D_1$, then either $h^2D_1 \subseteq D_1$ or $D_1 \cup h^2D_1 = S$. If the former occurs, it follows that $h^2J = J$, contrary to our assumption that $h^2J \neq J$; and if the latter occurs, $S \cap \text{fix}(h^2) = \emptyset$, again a contradiction. Hence, $h^2J \subseteq \text{int}(D_2)$ and it is easy to verify that $h^2D_1 \subseteq D_2$. We let $S' = D \cup A \cup h^2D$, where $A = \text{Cl}(S - (D_1 \cup h^2D_1))$. Clearly, $h^2S' = S'$ and by construction $S' \cap hS' = A \cap hA$ is a strict subset of $S \cap hS$.

Case b. Suppose $hJ = J$. We claim that $J \cap \text{fix}(h^2) = \emptyset$. Suppose not. Then $J \cap \text{fix}(h^2)$ consists of one or two points. If $J \cap \text{fix}(h^2)$ consists of two points, then $h^2D = \text{Cl}(hS - D)$, i.e., $h^2D \neq D$ since $\text{int}(D) \cap \text{fix}(h^2) = \emptyset$. We thus have the following: $D \cup D_1$ bounds a 3-cell, and hence $h(D \cup D_1) = D_1 \cup \text{Cl}(hS - D)$ bounds a 3-cell. But if $D_1 \cup \text{Cl}(hS - D)$ bound a 3-cell, then $hS = D \cup \text{Cl}(hS - D)$ is homotopic to $D \cup D_1$, contrary to our choice of S. Hence, $J \cap \text{fix}(h^2)$ does not contain two points. But $\text{fix}(h^2)$ cannot consist of one point as demonstrated previously, and thus $J \cap \text{fix}(h^2) = \emptyset$.

Since $J \cap \text{fix}(h^2) = \emptyset$, it follows that $D \cap \text{fix}(h^2) \neq \emptyset$ and $D_2 \cap \text{fix}(h^2) \neq \emptyset$. Therefore, $h^2D = D$ and $h^2D_2 = D_2$. We choose a simple closed curve $J' \subseteq D_2$ sufficiently close to J so that the annulus, A, bounded by

J and J' has the property that $A \cap hS = J$. Now a disk, D', is chosen sufficiently close to D such that $h^2D' = D'$, $D' \cap hD' = D' \cap hS = \emptyset$, $D' \cap S = \partial D' = J'$, and $S' = (D_2 - A) \cup D'$ does not bound a 3-cell. One may verify that $h^2S' = S'$ and $S' \cap hS' = (D_2 - A) \cap h(D_2 - A)$ is a strict subset of $S \cap hS$.

By repeating the above alterations a finite number of times, we can construct a 2-sphere, S, such that S does not bound a 3-cell and $hS = S$ or $S \cap hS = \emptyset$ with the property that $h^2S = S$.

CHAPTER THREE
PERIOD FOUR ACTIONS ON $S^2 \times (0,1)$

In this chapter, all Z_4 actions on $S^2 \times (0,1)$ are classified up to conjugation. We first describe the standard Z_4 actions on $S^2 \times (0,1)$. View S^2 as a subset of R^3 in the following sense:

$$S^2 = \{(x_1,x_2,x_3) \in R^3 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}.$$

Let $s_1\colon S^2 \to S^2$ be the standard semifree Z_4 action on S^2 defined by $s_1(x_1,x_2,x_3) = (-x_2,x_1,-x_3)$, and $s_2\colon S^2 \to S^2$ be the standard period 4 rotation defined by $s_2(x_1,x_2,x_3) = (-x_2,x_1,x_3)$. Let $\lambda_1\colon (0,1) \to (0,1)$ denote the identity, and $\lambda_2\colon (0,1) \to (0,1)$ be defined by $\lambda_2(t) = 1-t$, $t \in (0,1)$. Then $s_1 \times \lambda_i$, $i = 1,2$, are the standard semifree Z_4 actions, and $s_2 \times \lambda_i$, $i = 1,2$, are the standard nonfree Z_4 actions. We show that $S^2 \times (0,1)$ admits exactly four Z_4 actions up to conjugation.

Z_4 Actions on $S^2 \times [0,1]$

In this section we prove that $S^2 \times [0,1]$ admits exactly four nonequivalent Z_4 actions up to conjugation. We first classify the semifree Z_4 actions, then the nonfree Z_4 actions, and finally show that $S^2 \times [0,1]$ admits no free Z_4 actions. The method of proof in all cases but

one will be to extend h in a natural way to a Z_4 action $h'\colon S^3 \to S^3$ and use the classification theorems of Waldhausen [21], and Kim [8], [9], for Z_4 actions on S^3 to obtain information about the homeomorphism h.

The following result will be needed to classify the semifree Z_4 actions on $S^2 \times [0,1]$.

Lemma 3.1. Let $h\colon S^2 \times [0,1] \to S^2 \times [0,1]$ be a semi-free Z_4 action on $S^2 \times [0,1]$ such that $h(S^2 \times i) = S^2 \times (1-i)$, $i = 0,1$. Then there is a 2-sphere $S_0 \subseteq S^2 \times (0,1)$ such that S_0 does not bound a 3-cell and $hS_0 = S_0$.

Proof of Lemma 3.1.

We attach two 3-balls, B_0 and B_1, to $S^2 \times [0,1]$ via homeomorphisms $\gamma_0\colon B_0 \to S^2 \times 0$ and $\gamma_1\colon B_1 \to S^2 \times 1$. Then $B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1$ is homeomorphic to S^3, and h can be extended in the obvious way to obtain a homeomorphism $h'\colon [B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1] \to [B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1]$ with $\mathrm{fix}(h') = \emptyset$ and $\mathrm{fix}(h'^2)$ homeomorphic to S^1. By Theorem C [8], h' is conjugate to the standard action $s'\colon S^3 \to S^3$ defined by $s'(x_1,x_2,x_3,x_4) = (-x_2,x_1,-x_3,-x_4)$. Hence, there is an h'-invariant 2-sphere, S_0, such that $S_0 \cap \mathrm{fix}(h')$ consists of two points. If $S_0 \subseteq S^2 \times (0,1)$, then S_0 does not bound a 3-cell in $S^2 \times [0,1]$. Otherwise, $h|_B\colon B \to B$ is a fixed point free homeomorphism, where B is the 3-cell bounded by S_0, a contradiction. Thus, S_0 satisfies the conclusion of the Lemma. Therefore, assume $S_0 \not\subseteq S^2 \times (0,1)$.

We will alter S_0 to a 2-sphere S_0' which satisfies the conclusion of the lemma.

If $S_0 \not\subseteq S^2 \times (0,1)$, then $S_0 \cap (S^2 \times 0 \cup S^2 \times 1) \neq \emptyset$. Otherwise, $S_0 \subseteq B_i$ for $i = 1$ or $i = 2$ contradicting the fact that S_0 is h'-invariant. By using general positioning techniques similar to those used in the proof of Lemma 2.2, we can alter S_0 to a 2-sphere, S_0', such that S_0' is invariant under h' and either $S_0' \cap S^2 \times 0 = \emptyset$ or $S_0' \cap S^2 \times 0$ consists of a finite number of simple closed curves at which S_0' and $S^2 \times 0$ meet transversely. For simplicity let $S_0' = S_0$. If $S_0 \cap S^2 \times 0 = \emptyset$ then $h(S_0 \cap S^2 \times 0) = S_0 \cap S^2 \times 1 = \emptyset$ and $S_0 \subseteq S^2 \times (0,1)$. Therefore, assume $S_0 \cap S^2 \times 0 \neq \emptyset$ and let D be an innermost disk on $S^2 \times 0$ with boundary $J \subseteq S_0 \cap S^2 \times 0$. It follows easily that $J \cap \text{fix}(h^2) = \emptyset$, and since D is innermost either $h^2 D = D$ or $D \cap h^2 D = \emptyset$. J separates S_0 into two components whose closures, D_1 and D_2, are disks with $\partial D_1 = \partial D_2 = J$. Because S_0 and $S^2 \times 0$ meet transversely at J, D_i satisfies the property that $A' \cap S^2 \times (0,1) \neq \emptyset$ for every annulus $A' \subseteq D_i$, where $\partial A' = J \cup K$ and $K \subseteq \text{int } D_i$, $i = 1$ or $i = 2$. Without loss of generality, assume that D_1 satisfies this property. Since $h(S^2 \times i) = S^2 \times (1-i)$, $i = 0,1$, $hJ \subseteq S^2 \times 1$ and hence $hJ \neq J$. We now have two cases to consider.

Case 1. Suppose $h^2 D = D$. Then $h'^2 D_i = D_i$, $i = 1,2$, and either $hJ \subseteq D_1$ of $hJ \subseteq D_2$.

a. If $hJ \subseteq D_1$, then $h'D_2 \subseteq D_1$. Otherwise, $D_2 \subseteq h'D_2$, and since $h'^2 D_2 = D_2$, it follows that $h'D_2 = D_2$

and $hJ = J$, a contradiction. First, let $S_0^* = C \cup D \cup hD$, where $C = Cl(S_0 - D_2 \cup h'D_2)$. Now choose a simple closed curve $J' \subseteq C$ sufficiently close to J so that the annulus, A, bounded by J and J' has the property that $A \cap S^2 \times 0 = J$ and $h'^2A = A$. Choose a disk $D' \subseteq S^2 \times (0,1)$ sufficiently close to D such that $h'^2D' = D'$, $D' \cap h'D' = \emptyset$, and $D' \cap S_0^* = \partial D' = J'$. Let

$$S_0' = [(C - (A \cup h'A)) \cup (D' \cup h'D')].$$

S_0' is clearly invariant under h' and $S_0' \cap S^2 \times 0$ is a strict subset of $S_0 \cap S^2 \times 0$.

b. If $hJ \subseteq D_2$, it follows by an argument similar to the above that $h'D_1 \subseteq D_2$. In this case, we first let $S_0^* = E \cup D \cup hD$, where $E = Cl(S_0 - (D_1 \cup h'D_1))$. Now choose a simple closed curve $J' \subseteq E$ sufficiently close to J such that the annulus, A, bounded by J and J' has the properties that $A \cap S^2 \times 0 = J$ and $h'^2A = A$. Choose a disk $D' \subseteq int(B_0)$, where $\partial B_0 = S^2 \times 0$, sufficiently close to D such that $h'^2D' = D'$, $D' \cap h'D' = \emptyset$, and $D' \cap S_0^* = \partial D' = J'$. Let

$$S_0' = [(E - (A \cup h'A)) \cup (D' \cup h'D')].$$

S_0' is invariant under h' and $S_0' \cap S^2 \times 0$ is a strict subset of $S_0 \cap S^2 \times 0$.

Case 2. Suppose $D \cap h^2D = \emptyset$. In particular, $h^2J \neq J$. Again, let D_1 and D_2 be defined as in case a. Now either $D_1 \cap h'^2D_1 = \emptyset$ or $D_1 \cap h'^2D_1 = A$, where $A_0 \subseteq S_0$ is the

annulus bounded by $J \cup h^2 J$ and $S_0 \cap \text{fix}(h'^2) \subseteq A$. Moreover, if $D_1 \cap h'^2 D_1 = \emptyset$, S_0 has a decomposition into $S_0 = D_2 \cup A \cup h'^2 D_2$.

a. Suppose $D_1 \cap h'^2 D_1 = \emptyset$. Since $h(S^2 \times i) = S^2 \times (1 - i)$, $i = 0,1$, it follows that $h'^j D_1 \cap h'^k D_1 = \emptyset$ $j \neq k$, $1 \le j \le 4$, $1 \le k \le 4$. First let

$$S_0^* = [(S_0 - (\bigcup_{1 \le j \le 4} h;^j (D_1))) \cup (\bigcup_{1 \le j \le 4} h^j D)].$$

Clearly S_0^* is invariant under h'. Now let J' be a simple closed curve on $S^0 - (\bigcup_{1 \le j \le 4} h'^j (D_1))$ sufficiently close to J such that the annulus, A, bounded by J and J' has the property that $h^k A \cap (S^2 \times 0 \cup S^2 \times 1) = h^k J$, $1 \le k \le 4$. Now choose a disk $D' \subseteq S^2 \times (0,1)$ sufficiently close to D so that $h^j D' \cap h^k D' = \emptyset$ for $j \neq k$, $1 \le j \le 4$ and $1 \le k \le 4$, and $D' \cap S_0^* = \partial D' = J'$. Let

$$S_0' = [(S_0^* - (\bigcup_{1 \le k \le 4} h^k D)) \cup (\bigcup_{1 \le k \le 4} h^k D')].$$

S_0' is clearly invariant under h' and $S_0' \cap S^2 \times 0$ is a strict subset of $S_0 \cap S^2 \times 0$.

b. Suppose $D_1 \cap h^2 D_1 = A$. We repeat the construction as in case 2a, with the exception that D_2 replaces D_1 in each formula, and the disk, D', is chosen as a subset of B_0. The details follow exactly the methods in case a, with the above noted exceptions, and thus are omitted.

By repeating the above alterations a finite number of times we can construct a 2-sphere, S_0, such that $S_0 \subseteq S^2 \times (0,1)$ and $hS_0 = S_0$.

The following lemma classifies the semifree Z_4 actions on $S^2 \times [0,1]$.

Lemma 3.2. Let $h\colon S^2 \times [0,1] \to S^2 \times [0,1]$ be a semifree Z_4 action on $S^2 \times [0,1]$. Then there is a homeomorphism $\alpha\colon S^2 \times [0,1] \to S^2 \times [0,1]$ such that $\alpha h \alpha^{-1}(x,t) = (s_1(x), \lambda(t))$, $(x,t) \in S^2 \times [0,1]$, where $s_1\colon S^2 \to S^2$ is the standard semifree Z_4 action on S^2 and $\lambda(t) = t$ or $\lambda(t) = 1-t$.

Proof of Lemma 3.2.

There are two cases to consider.

Case 1. Suppose $h(S^2 \times i) = S^2 \times i$, $i = 0,1$. We first attach two 3-balls, B_0 and B_1, via homeomorphisms $\gamma_0\colon B_0 \to S^2 \times 0$ and $\gamma_1\colon B_1 \to S^2 \times 1$. Then $B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1$ is homeomorphic to S^3, and h can be extended in the obvious way to obtain a homeomorphism $h'\colon [B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1] \to [B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1]$ with $\mathrm{fix}(h')$ homeomorphic to S^0 and $\mathrm{fix}(h'^2)$ homeomorphic to S^1. We view S^3 as a subset of R^4 in the following sense:

$$S^3 = \{(x_1,x_2,x_3,x_4) \in R^4 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1\}.$$

By Theorem 2.2 [9], there is a homeomorphism $\alpha'\colon [B_0 \underset{\gamma_0}{\cup} S^2 \times [0,1] \underset{\gamma_1}{\cup} B_1] \to S^3$ such that $\alpha' h' \alpha'^{-1} = s$, where $s\colon S^3 \to S^3$ is defined by $s(x_1,x_2,x_3,x_4) = (-x_2,x_1,-x_3,x_4)$. By the construction of the conjugation map α' (see Theorem 2.2 [9]), we may assume that $\alpha'(S^2 \times [0,1]) \subseteq C$ where

$$C = \{(x_1,x_2,x_3,x_4) \in S^3 \mid -1/2 \le x_4 \le 1/2\}.$$

We also consider $S^2 \times [0,1]$ as a subset of R^4 in the following manner:

$$S^2 \times [0,1] = \{(x_1,x_2,x_3,t) \in R^4 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 = 1 \text{ and } 0 \le t \le 1\}.$$

Now define $\gamma\colon C \to S^2 \times [0,1]$ by $\gamma(x_1,x_2,x_3,x_4) = (x_1/r, x_2/r, x_3/r, (2t+1)/2)$, where $r = (1 - (x_4)^2)^{1/2}$, and set $\alpha = \gamma\alpha' \mid_{S^2 \times [0,1]}$. Then $\alpha h \alpha^{-1}(x,t) = (s_1(x),t)$.

Case 2. Suppose $h(S^2 \times i) = S^2 \times (1 - i)$, $i = 0,1$. By attaching two 3-balls, B_0 and B_1, via homeomorphisms $\gamma_0\colon B_0 \to S^2 \times 0$ and $\gamma_1\colon B_1 \to S^2 \times 1$ to $S^2 \times [0,1]$ and extending h in the obvious way, we obtain a homeomorphism $h'\colon [B_0 \cup_{\gamma_0} S^2 \times [0,1] \cup_{\gamma_1} B_1] \to [B_0 \cup_{\gamma_0} S^2 \times [0,1] \cup_{\gamma_1} B_1]$ with $\text{fix}(h') = \emptyset$ and $\text{fix}(h'^2)$ homeomorphic to S^1. Thus $\text{fix}(h^2)$ is homeomorphic to two closed arcs. By Lemma 3.1, there is a 2-sphere $S_0 \subseteq S^2 \times (0,1)$ such that S_0 does not bound a 3-cell and $hS_0 = S_0$. It follows from Theorem 2.2 [9] that there is a homeomorphism $\alpha_0\colon S_0 \to S^2 \times 0$ such that $\alpha_0 h \alpha_0^{-1}(x,t) = (s_1(x),0)$ for $(x,t) \in S_0$. Now S_0 divides $S^2 \times [0,1]$ into two components whose closures are 3-annuli, say A_1 and A_2 (see [3]). Assume that A_1 is bounded by S_0 and $S^2 \times 1$ in $S^2 \times [0,1]$. A_1 is invariant under h^2 and $hA_1 = A_2$. Since $\text{fix}(h^2)$ is homeomorphic to two closed arcs and $S_0 \cup \text{fix}(h^2)$ is two points, $A_1 \cup \text{fix}(h^2)$ is also homeomorphic to two closed arcs. It follows easily from

Theorem 1.2 that there is a homeomorphism $\alpha_1 \colon A_1 \to S^2 \times [0,1]$ such that $\alpha_1 h^2 \alpha_1^{-1}(x,t) = (s_1^{\,2}(x),t)$, and $\alpha_1 S_0 = S^2 \times 0$.

We now alter α_1 to a homeomorphism $\alpha_1^* \colon A_1 \to S^2 \times [0,1]$ such that $\alpha_1^*(x,t) = \alpha_0(x,t)$ for $(x,t) \in S_0$ and $\alpha_1^* h^2 \alpha_1^{*-1}(x,t) = (s_1^{\,2}(x),t)$. Define $H\colon S^2 \times [0,1] \to S^2 \times [0,1]$ by $H(x,t) = (p\alpha_0\alpha_1^{-1}(x,0),t)$, where $p\colon S^2 \times [0,1] \to S^2$ is defined by $p(x,t) = x$ and set $\alpha_1^* = H\alpha_1$. If $(x,t) \in S_0$, then $\alpha_1(x,t) = (x',0)$ for some $x' \in S^2$ and hence

$$\alpha^*_1(x,t) = H\alpha_1(x,t) = (p\alpha_0\alpha_1^{-1}(\alpha_1(x,t)),0)$$

$$= (p\alpha_0(x,t),0) = \alpha_0(x,t).$$

Next we verify that $\alpha_1^* h^2 \alpha_1^{*-1}(x,t) = (s_1^{\,2}(x),t)$, $(x,t) \in S^2 \times [0,1]$. By definition $\alpha_1^* h^2 \alpha_1^{*-1} = H\alpha_1 h^2 \alpha_1^{-1} H^{-1}$, and since $\alpha_1 h^2 \alpha_1^{-1}(x,t) = (s_1^{\,2}(x),t)$, it suffices to show that $Hs^*H^{-1}(x,t) = (s_1^{\,2}(x),t)$, where $s^*(x,t) \equiv (s_1^{\,2}(x),t)$. But $H^{-1}(x,t) = (p\alpha_1\alpha_0^{-1}(x,0),t)$ and thus,

$$\begin{aligned}
Hs^*H^{-1}(x,t) &= H(s^*(p\alpha_1\alpha_0^{-1}(x,0),t)) \\
&= H(s_1^{\,2}(p\alpha_1\alpha_0^{-1}(x,0)),t) \\
&= (p\alpha_0\alpha_1^{-1}(s_1^{\,2}(p\alpha_1\alpha_0^{-1}(x,0)),0),t) \\
&= (p\alpha_0\alpha_1^{-1}s^*\alpha_1\alpha_0^{-1}(x,0),t) \\
&= (p\alpha_0 h^2 \alpha_0^{-1}(x,0),t) \\
&= (s_1^{\,2}(x),t).
\end{aligned}$$

Thus α_1^* is the desired homeomorphism.

Now $S^2 \times [0,1] = A_1 \cup hA_1$, and we define $\alpha^*: S^2 \times [0,1] \to S^2 \times [-1,1]$ by $\alpha^*(x,t) = \alpha_1^*(x,t)$ for $(x,t) \in A_1$ and $\alpha^*(h(x,t)) = s\alpha_1^*(x,t)$ for $h(x,t) \in hA_1$, where $s: S^2 \times [-1,1] \to S^2 \times [-1,1]$ is defined by $s(x,t) = (s_1(x),-t)$. We claim that $\alpha^* h \alpha^{*-1} = s$. If $(x,t) \in A_1$ then $h(x,t) = \alpha^{*-1} s\alpha^*(x,t)$ by definition. If $h(x,t) \in hA$, then $\alpha^{*-1} s\alpha^*(h(x,t)) = \alpha^{*-1} s^2 \alpha_1^*(x,t) = \alpha_1^{*-1} s^2 \alpha_1^*(x,t)$. But $\alpha_1^{*-1} s^2 \alpha_1^*(x,t) = h^2(x,t)$, $(x,t) \in A_1$, by the choice of α_1^* and hence $h(h(x,t)) = \alpha^{*-1} s\alpha^*(h(x,t))$, $h(x,t) \in A_1$. Thus, we have a homeomorphism $\alpha^*: S^2 \times [0,1] \to S^2 \times [-1,1]$ such that $\alpha^* h \alpha^{*-1}(x,t) = (s_1(x),-t)$. We finally define $\tau: S^2 \times [-1,1] \to S^2 \times [0,1]$ by $\tau(x,t) = (x,(t+1)/2)$, and set $\alpha = \tau\alpha^*$. Then $\alpha h \alpha^{-1}(x,t) = (s_1(x),1-t)$. This completes the proof of Lemma 3.2.

We now turn to the nonfree Z_4 actions on $S^2 \times [0,1]$. Let $h: S^2 \times [0,1] \to S^2 \times [0,1]$ be a nonfree Z_4 action. By attaching two 3-balls to $S^2 \times [0,1]$ and extending h in the obvious way, we obtain a homeomorphism $h': S^3 \to S^3$ with $\operatorname{fix}(h) \subseteq \operatorname{fix}(h')$. As noted in the proof of Theorem 2.1, $\operatorname{fix}(h'^2)$ is homeomorphic to S^1. Since $\operatorname{fix}(h')$ is an i-sphere, $i = 0,1,2$ (see [18]), and $\operatorname{fix}(h') \subseteq \operatorname{fix}(h'^2)$, it follows that $\operatorname{fix}(h')$ is homeomorphic to S^0 or S^1. Thus, $\operatorname{fix}(h)$ is homeomorphic to S^0 or $\operatorname{fix}(h) = \operatorname{fix}(h^2)$ is homeomorphic to two closed arcs. The following two lemmas classify the nonfree Z_4 actions.

Lemma 3.3. Let $h: S^2 \times [0,1] \to S^2 \times [0,1]$ be a Z_4 action on $S^2 \times [0,1]$ with fix(h) homeomorphic to S^0. Then there is a homeomorphism $\alpha: S^2 \times [0,1] \to S^2 \times [0,1]$ such that $\alpha h \alpha^{-1}(x,t) = (s_2(x), 1-t)$, where $s_2: S^2 \to S^2$ is the standard period 4 rotation on S^2 leaving the poles fixed.

Proof of Lemma 3.3.

We attach two 3-balls to $S^2 \times [0,1]$ and extend h in the obvious way to obtain a homeomorphism, $h': S^3 \to S^3$, with fix (h') homeomorphic to S^0 and $\text{fix}(h'^2)$ homeomorphic to S^1. By Theorem 2.2 [9], h' is conjugate to the standard action, $s: S^3 \to S^3$, defined by $s(x_1,x_2,x_3,x_4) = (-x_2,x_1,x_3,-x_4)$. Thus, there is a 2-sphere, S_0, such that $hS_0 = S_0$ and $S_0 \cap \text{fix}(h') = S_0 \cap \text{fix}(h'^2) \approx S^0$. Moreover, by using a general positioning argument similar to that used in the proof of Lemma 3.1, we may assume that $S_0 \subseteq S^2 \times (0,1)$. Also, S_0 does not bound a 3-cell in $S^2 \times [0,1]$. Otherwise $h|_B: B \to B$ is a homeomorphism such that $\text{fix}(h|_B)$ is two points and $h|_B$ is periodic, where B is the 3-cell with $\partial B = S_0$, a contradiction. It follows from Theorem 2.2 [9] that there is a homeomorphism $\alpha_0: S_0 \to S^2 \times 1/2$ such that $\alpha_0 h_0 \alpha_0^{-1}(x,t) = (s_2(x), 1/2)$, $(x,t) \in S_0$, where h_0 is the restriction of h to S_0. Now S_0 separates $S^2 \times [0,1]$ into two components whose closures, A_1 and A_2, are 3-annuli. Since fix(h) is two points, $\text{fix}(h^2)$ is homeomorphic to two closed arcs, and $S_0 \cap \text{fix}(h) = S_0 \cap \text{fix}(h^2) \approx S^0$, it follows that $hA_1 = A_2$, A_1 is invariant under h^2, and $A_1 \cap \text{fix}(h^2)$ is homeomorphic to two closed

arcs. Hence, by Theorem 1.2, there is a homeomorphism $\alpha_1: A_1 \to S^2 \times [1/2,1]$ such that $\alpha_1 h^2 \alpha_1^{-1}(x,t) = (s_2{}^2(x),t)$. By the same argument as that in Lemma 3.2, we may choose α_1 so that $\alpha_1(x,t) = \alpha_0(x,t)$ for $(x,t) \in S_0$. Define $s^*: S^2 \times [0,1] \to S^2 \times [0,1]$ by $s^*(x,t) = (s_2(x),1-t)$. Let $\alpha: S^2 \times [0,1] \to S^2 \times [0,1]$ be defined by $\alpha(x,t) = \alpha_1(x,t)$ for $(x,t) \in A_1$, and $\alpha(h(x,t)) = s^*\alpha_1(x,t)$ for $h(x,t) \in hA_1$. Using the techniques in Lemma 3.2, one may verify that $\alpha h \alpha^{-1}(x,t) = (s_2(x),1-t)$ and the lemma is established.

Lemma 3.4. Let $h: S^2 \times [0,1] \to S^2 \times [0,1]$ be a Z_4 action on $S^2 \times [0,1]$ with a $\mathrm{fix}(h^k)$ homeomorphic to two closed arcs, $1 \le k < 4$. Then there is a homeomorphism $\alpha: S^2 \times [0,1] \to S^2 \times [0,1]$ such that $\alpha h \alpha^{-1}(x,t) = (s_2(x),t)$, where $s_2: S^2 \to S^2$ is the standard period 4 rotation on S^2 leaving the poles fixed.

Proof of Lemma 3.4.

For simplicity of notation, let $s: S^2 \times [0,1] \to S^2 \times [0,1]$ be defined by $s(x,t) = (s_2(x),t), (x,t) \in S^2 \times [0,1]$. Since h^2 is an involution and $\mathrm{fix}(h^2)$ is homeomorphic to two closed arcs, it follows easily from Theorem 1.2 that h^2 is conjugate to s^2 and hence $(S^2 \times [0,1])/\langle h^2 \rangle$ is homeomorphic to $S^2 \times [0,1]$. Now h induces an involution $h^*: (S^2 \times [0,1])/\langle h^2 \rangle \to (S^2 \times [0,1])/\langle h^2 \rangle$ which makes the following diagram commute:

$$\begin{array}{ccc} S^2 \times [0,1] & \xrightarrow{h} & S^2 \times [0,1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ (S^2 \times [0,1])/\langle h^2\rangle & \xrightarrow{h^*} & (S^2 \times [0,1])/\langle h^2\rangle, \end{array}$$

where $p\colon S^2 \times [0,1] \to (S^2 \times [0,1])/\langle h^2\rangle$ is the natural quotient map. Also s induces the standard involution $s^*\colon (S^2 \times [0,1]/\langle s^2\rangle \to (S^2 \times [0,1])/\langle s^2\rangle$ so that $p_1 s = s^* p_1$, where $p_1\colon S^2 \times [0,1] \to (S^2 \times [0,1])/\langle s^2\rangle$ is the natural quotient map. Moreover, $p|_{\text{fix}(h)}\colon \text{fix}(h) \to \text{fix}(h^*)$ is a homeomorphism and thus fix(h*) is homeomorphic to two closed arcs. Again applying Theorem 1.2 to the homeomorphism h*, there is a homeomorphism $\alpha'\colon (S^2 \times [0,1])/\langle h^2\rangle \to (S^2 \times [0,1])/\langle s^2\rangle$ such that $\alpha' h^* \alpha'^{-1} = s^*$. Since

$$\alpha'_* p_* (\pi_1(S^2 \times [0,1] - \text{fix}(h))) = p_{1*}(\pi_1(S^2 \times [0,1] - \text{fix}(s))),$$

by the Lifting Theorem, there is a homeomorphism $\alpha\colon [S^2 \times [0,1] - \text{fix}(h)] \to [S^2 \times [0,1] - \text{fix}(s)]$ such that $\langle \alpha h \alpha^{-1}\rangle = \langle s|_{S^2 \times [0,1] - \text{fix}(s)}\rangle$. Now $p|_{\text{fix}(h)}\colon \text{fix}(h) \to \text{fix}(h^*)$ and $p_1|_{\text{fix}(s)}\colon \text{fix}(s) \to \text{fix}(s^*)$ are homeomorphisms, and α may be extended to fix(h) by defining $\alpha(x,t) = p_1^{-1}\alpha' p(x,t), (x,t) \in \text{fix}(h)$. Thus, there is a homeomorphism $\alpha\colon S^2 \times [0,1] \to S^2 \times [0,1]$ such that $\langle \alpha h \alpha^{-1}\rangle = \langle s\rangle$. But since s and s^3 are conjugate, we have that h is conjugate to s.

Although Lemma 3.4 will not be needed in the sequel, it together with Lemmas 3.2 and 3.3 provide a classification

of all Z_4 actions on $S^2 \times [0,1]$. For if $h: S^2 \times [0,1] \to S^2 \times [0,1]$ is a Z_4 action, by attaching two 3-balls, B_0 and B_1, and extending h to a homeomorphism $h': S^3 \to S^3$, $\text{fix}(h'^2)$ is homeomorphic to S^1 and $\text{fix}(h'^2) \cap B_i \neq \emptyset$, $i = 0,1$. Therefore, if $\text{fix}(h^2) = \emptyset$ then $S^2 \times [0,1]$ separates $\text{fix}(h'^2)$ in S^3, a contradiction. Thus, there are no free Z_4 actions on $S^2 \times [0,1]$ and we have the following corollary.

Corollary. $S^2 \times [0,1]$ admits exactly four nonequivalent Z_4 actions up to conjugation.

Z_4 Actions on $S^2 \times (0,1)$

In this section we show that $S^2 \times (0,1)$ admits exactly four nonequivalent Z_4 actions up to conjugation. Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a Z_4 action on $S^2 \times (0,1)$, and let $D = [S^2 \times (0,1) \cup \{-\infty,\infty\}]$ denote the two point compactification of $S^2 \times (0,1)$. Then D is homeomorphic to S^3 and there is a homeomorphism (not necessarily PL), $h^*: D \to D$ which extends h. Clearly h^{*2} is orientation preserving and $\text{fix}(h^{*2}) \neq \emptyset$ since $\{-\infty,\infty\} \subseteq \text{fix}(h^{*2})$. It follows from Smith's Theorem [18] that $\text{fix}(h^{*2})$ is homeomorphic to S^1. Now, either $\text{fix}(h^*)$ is empty, in which case $\text{fix}(h) = \emptyset$, or $\text{fix}(h^*)$ is homeomorphic to S^i, $i = 0,1$, or 2. Since $\text{fix}(h^*) \subseteq \text{fix}(h^{*2})$ it follows that $\text{fix}(h^*)$ is homeomorphic to S^0 or S^1. If $\text{fix}(h^*)$ is homeomorphic to S^0 and $\text{fix}(h^*) = \{-\infty,\infty\}$, then $\text{fix}(h) = \emptyset$; otherwise

$\text{fix}(h^*) \cap \{-\infty,\infty\} = \emptyset$ and fix(h) is also homeomorphic to S^0. If fix(h*) is homeomorphic to S^1, then $\text{fix}(h^*) = \text{fix}(h^{*2})$ and $\text{fix}(h) = [\text{fix}(h^*) - \{-\infty,\infty\}]$ is homeomorphic to two open arcs. Thus, fix(h) is either empty, homeomorphic to S^0, or homeomorphic to two open arcs. We consider each of these cases separately.

Theorem 3.5. Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a semifree Z_4 action. Then there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\alpha h \alpha^{-1}(x,t) = (s_1(x), \lambda(t))$, where $x \in S^2$, $t \in (0,1)$; $\lambda(t) = t$ or $\lambda(t) = 1-t$; and $s_1: S^2 \to S^2$ is the standard semifree Z_4 action on S^2.

Proof of Theorem 3.5.

By Lemma 2.3, there is a 2-sphere, S_0, such that $hS_0 = S_0$ or $S_0 \cap hS_0 = \emptyset$ with $h^2S_0 = S_0$, and S_0 does not bound a 3-cell. We have two cases to consider.

Case 1. Suppose $hS_0 = S_0$. S_0 separates $S^2 \times (0,1)$ into two components, A_1 and B_1, each homeomorphic to the interior of a 3-annulus, and either $hA_1 = A_1$ or $hA_1 = B_1$.

a. If $hA_1 = A_1$, then A_1 is invariant under h, and we apply Lemma 2.3 again to find a 2-sphere, S_1, in A_1 such that S_1 does not bound a 3-cell and either $hS_1 = S_1$ or $S_1 \cap hS_1 = \emptyset$ with $h^2S_1 = S_1$. But $\text{Cl}(A_1)$ is invariant under h, and therefore $hS_1 \cap S_1 = \emptyset$ cannot occur. Thus, $hS_1 = S_1$. Now S_1 separates A_1 into two components, A_2 and B_2, each homeomorphic to the interior of a 3-annulus, and since $\text{Cl}(A_1)$ is invariant under h, $hA_2 = A_2$. If $S^2 \times (0,1)$ is

viewed as a subset of $S^2 \times [0,1]$, assume A_2 is bounded by S_1 and $S^2 \times 0$ in $S^2 \times [0,1]$. Again there is a 2-sphere $S_2 \subseteq A_2$ such that S_2 does not bound a 3-cell and $hS_2 = S_2$. Continuing in this manner, we construct a sequence, $\{S_n\}_{n=1}^{\infty}$ of h-invariant 2-spheres such that $S_n \subseteq A_n$, where A_n is the interior of the 3-annulus bounded by $S^2 \times 0$ and S_{n-1} in $S^2 \times [0,1]$, and S_n does not bound a 3-cell in A_n. We may also require that the sequence $\{S_n\}_{n=1}^{\infty}$ satisfy the following property: given any collar $C = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$, there is an n, $n \geq 1$, such that $S_n \subseteq C$. To obtain this additional restriction, let $\{C_n\}_{n=1}^{\infty}$ be the sequence of collars of $S^2 \times 0$ in $S^2 \times [0,1]$ defined by $C_n = S^2 \times [0,1/2^n]$, $n \geq 1$. We choose S_0 as before. Since $Cl(A_1)$ is invariant under h, one may easily verify that given any collar $C = S^2 \times [0,c]$ of $S^2 \times 0$, $S^2 \times (0,c] \cap h(S^2 \times (0,c]) \neq \emptyset$, and it follows that there is a t, $0<t<c$, such that $h(S^2 \times (0,t]) \subseteq C$. Now choose an $n(1)$, $n(1) \geq 1$ such that $C_{n(1)} \cap S_0 = \emptyset$, and t_1 and t_2, $0<t_2<t_1<1/2^{n(1)}$, satisfying $h(S^2 \times (0,t_1]) \subseteq C_{n(1)}$ and $h(S^2 \times (0,t_2]) \subseteq S^2 \times (0,t_1]$. By Theorem 2.1, there is a nontrivial 2-sphere, S_1^{**}, such that $h^2(S_1^{**}) = S_1^{**}$, where $S_1^{**} \subseteq S^2 \times (0,t_2)$. Using Lemma 2.2, we alter S_1^{**} with arbitrarily small isotopies so that the resulting 2-sphere, S_1^*, satisfies the properties that $h^2 S_1^* = S_1^*$, $S_1^* \cap hS_1^* = \emptyset$ or $S_1^* \cap hS_1^*$ consists of a finite number of simple closed curves at which S_1^* and hS_1^* meet transversely, and

$S_1^* \subseteq S^2 \times (0,t_1)$. Since $Cl(A_1)$ is invariant under h, $S_1^* \cap hS_1^*$ cannot occur. By the choice of t_1, $hS_1^* \subseteq int(C_{n(1)})$. Applying Lemma 2.3 to alter S_1^*, all alterations are made in $int(C_{n(1)})$ to obtain a 2-sphere, S_1, such that S_1 does not bound a 3-cell, $hS_1 = S_1$ and $S_1 \subseteq C_{n(1)} \subseteq A_1$. Now there is an $n(2) > n(1)$ such that $C_{n(2)} \cap S_1 = \emptyset$. To construct S_2, the above procedure is repeated with $C_{n(1)}$ replaced by $C_{n(2)}$. We continue in this manner to produce the sequence $\{S_n\}_{n=1}^{\infty}$. The above process is duplicated in B_1, where $S^2 \times (0,1) - S_0 = A_1 \cup B_1$, to obtain another sequence of h-invariant 2-spheres, $\{S_n\}_{n=-1}^{-\infty}$ such that $S_n \subseteq D_n$, where D_n is the interior of the 3-annulus bounded by S_{n+1} and $S^2 \times 1$ in $S^2 \times [0,1]$; S_n does not bound a 3-cell; and given any collar, D, of $S^2 \times 1$ there is an n, $n \leq -1$, such that $S_n \subseteq D$.

Now, let A_n^* denote the 3-annulus bounded by S_{n-1} and S_n, $n \geq 1$. Each A_n^* is invariant under h and $h_n\colon A_n^* \to A_n^*$ is a semifree Z_4 action, where h_n is the restriction of h to A_n^*. Hence, by Lemma 3.2, there is a homeomorphism $\beta_n\colon A_n^* \to S^2 \times [0,1]$ such that $\beta_n h_n \beta_n^{-1}(x,t) = (s_1(x),t)$, where $x \in S^2$, $t \in [0,1]$ and $s_1\colon S^2 \to S^2$ is the standard semifree Z_4 action on S^2. We may also assume that for each $n \geq 1$, $\beta_n(S_{n-1}) = S^2 \times 0$. Define, for each $n \geq 1$, $\delta_n\colon S^2 \times [0,1] \to S^2 \times [(n-1)/n, n/(n+1)]$ by $\delta_n(x,t) = (x,(n-1)/n+t/n(n+1))$, and set $\alpha_n = \delta_n\beta_n$. Then $\alpha_n h_n \alpha_n^{-1}(x,t) = (s_1(x),t)$, where $(x,t) \in S^2 \times [(n-1)/n, n/(n+1]$.

Next, we alter each α_n to a homeomorphism $\alpha_n^*: A_n^* \to S^2 \times [(n-1)/n, n/(n+1)]$ satisfying the following:

i. $\alpha_n^*(x,t) = \alpha_{n+1}^*(x,t)$ for $(x,t) \in S_n$, $n \geq 1$

ii. $\alpha_n^* h_n \alpha_n^{*-1}(x,t) = (s_1(x),t)$ for $(x,t) \in S^2 \times [(n-1)/n, n/(n+1)]$.

First let $\alpha_1^* = \alpha_1$. By using methods similar to those in Lemma 3.2, case 2, we construct α_2^* so that $\alpha_2^*(x,t) = \alpha_1^*(x,t)$ for $(x,t) \in S_1$, and $\alpha_2^* h_2 \alpha_2^{*-1}(x,t) = (s_1(x),t), (x,t) \in S^2 \times [1/2, 2/3]$. At the third stage α_3 is altered to a homeomorphism $\alpha_3^*: A_3^* \to S^2 \times [2/3, 3/4]$ so that $\alpha_3^*(x,t) = \alpha_2^*(x,t)$ for $(x,t) \in S_2$ and $\alpha_3^* h_3 \alpha_3^{*-1}(x,t) = (s_1(x),t)$, $(x,t) \in S^2 \times [2/3, 3/4]$. We continue in this manner to produce the sequence of homeomorphisms $\{\alpha_n^*\}_{n=1}^{\infty}$.

For each $n \leq -1$, let A_n^* denote the 3-annulus bounded by S_n and S_{n+1}. Then for each $n \leq -1$, there is also a homeomorphism $\alpha_n^*: A_n^* \to S^2 \times [-(n/(n-1)), -((n+1)/n)]$ such that $\alpha_n^* h_n \alpha_n^{*-1}(x,t) = (s_1(x),t)$, where

$$(x,t) \in S^2 \times [-(n/(n-1)), -((n+1)/n)]$$

and h_n is defined as above, and with the additional property that $\alpha_{-1}^*(x,t) = \alpha_1^*(x,t)$ for $(x,t) \in S_0$, and $\alpha_n^*(x,t) = \alpha_{n+1}^*(x,t)$, where $(x,t) \in S_{n+1}$ and $n \leq -2$.

By the construction of the sequence of 2-spheres, $\{S_n\}_{n=-\infty}^{\infty}$, $\cup_{n=-\infty}^{\infty} A_n = S^2 \times (0,1)$. Hence, define $\alpha^*: S^2 \times (0,1) \to S^2 \times (-1,1)$ by $\alpha^*(x,t) = \alpha_n^*(x,t)$ for $(x,t) \subseteq A_n^*$. Then α^* is a homeomorphism such that $\alpha^* h \alpha^{*-1}(x,t) = (s_1(x),t)$, where $(x,t) \in S^2 \times (-1,1)$. Finally

define δ: $S^2 \times (-1,1) \to S^2 \times (0,1)$ by $\delta(x,t) = (x,(t+1)/2)$ and set $\alpha = \delta\alpha^*$. It follows that $\alpha h \alpha^{-1}(x,t) = (s_1(x),t)$, where $(x,t) \in S^2 \times (0,1)$ and s_1: $S^2 \to S^2$ is the standard semifree Z_4 action on S^2.

b. Suppose $hA_1 = B_1$. Then $h^2 A_1 = A_1$, and it follows easily from Theorem 2.1 that there is a sequence, $\{S_n\}_{n=1}^{\infty}$, of nontrivial 2-spheres such that $h^2 S_n = S_n$; $S_n \subseteq A_n$, where A_n is the 3-annulus bounded by S_{n-1} and $S^2 \times 0$ in $S^2 \times [0,1]$; and given any collar $C = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$, there is an S_n so that $S_n \subseteq \mathrm{int}(C)$, $n \geq 1$. Since $hA_1 = B_1$, $\{hS_n\}_{n=1}^{\infty}$ is a sequence of nontrivial 2-spheres such that $h^2(hS_n) = hS_n$, and $hS_n \subseteq hA_n$, where hA_n is the annulus bounded by hS_{n-1} and $S^2 \times 1$ in $S^2 \times [0,1]$. The sequence $\{hS_n\}_{n=1}^{\infty}$ may also be shown to satisfy the additional property that given any collar, D, of $S^2 \times 1$ in $S^2 \times [0,1]$, there is an $n \geq 1$ such that $hS_n \subseteq D$. Let A_0^* denote the 3-annulus bounded by S_1 and hS_1. Then A_0^* is invariant under h, and by Lemma 3.2 there is a homeomorphism β_0: $A_0^* \to S^2 \times [0,1]$ such that $\beta_0 h_0 \beta_0^{-1}(x,t) = (s_1(x),1-t)$, where $(x,t) \in S^2 \times [0,1]$, h_0: $A_0^* \to A_0^*$ is the restriction of h to A_0^*, and s_1 is the standard semifree Z_4 action on S^2. Let ρ_0: $S^2 \times [0,1] \to S^2 \times [-1/2,1/2]$ be defined by $\rho_0(x,t) = (x,t-(1/2))$, and set $\alpha_0 = \rho_0\beta_0$. Then $\alpha_0 h_0 \alpha_0^{-1}(x,t) = (s_1(x),-t)$, where $(x,t) \in S^2 \times [-1/2,1/2]$. Let A_n^* denote the 3-annulus bounded by S_n and S_{n+1}, $n \geq 1$. A_n^* is invariant under h^2, and it follows easily from Theorem 2.1 that there is a homeomorphism β_n: $A_n^* \to S^2 \times [0,1]$

such that $\beta_n h_n{}^2 \beta_n{}^{-1}(x,t) = (s_1{}^2(x),t)$, where $h_n{}^2 \colon A_n^* \to A_n^*$ is the restriction of h^2 to A_n^*. Define

$$\rho_n \colon S^2 \times [0,1] \to S^2 \times [n/(n+1),(n+1)/(n+2)]$$

by $\rho_n(x,t) = (x, n/(n+1) + t/(n+1)(n+2))$, $n \geq 1$, and set $\alpha_n = \rho_n \beta_n$. Thus, $\alpha_n h_n{}^2 \alpha_n{}^{-1}(x,t) = (s_1{}^2(x),t)$, $(x,t) \in S^2 \times [n/(n+1),(n+1)/(n+2)]$. By an argument similar to that in case a, we may assume that the α_n's are chosen so that $\alpha_n(x,t) = \alpha_{n+1}(x,t)$, where $(x,t) \in S_n$, $n \geq 0$. Let $s \colon S^2 \times (-1,1) \to S^2 \times (-1,1)$ be defined by $s(x,t) = (s_1(x),-t)$. By construction, $S^2 \times (0,1) = (\cup_{n=0}^{\infty} A_n^*) \cup (\cup_{n=0}^{\infty} hA_n^*)$, and we define $\alpha^* \colon S^2 \times (0,1) \to S^2 \times (-1,1)$ by $\alpha^*(x,t) = \alpha_n(x,t)$, $(x,t) \in A_n^*$, and $\alpha^*(h(x,t)) = s\alpha_n(x,t)$, where $h(x,t) \in hA_n^*$.

We claim that $\alpha^* h \alpha^{*-1} = s$. If $(x,t) \in A_n^*$, then $h(x,t) = \alpha^{*-1} s \alpha^*(x,t)$ by definition. It remains to verify that $h(h(x,t)) = \alpha^{*-1} s \alpha^*(h(x,t))$ for $h(x,t) \in hA_n^*$. By definition, $\alpha^*(h(x,t)) = s\alpha_n(x,t)$ and thus

$$\alpha^{*-1} s \alpha^*(h(x,t)) = \alpha^{*-1} s^2 \alpha_n(x,t) = \alpha_n{}^{-1} s^2 \alpha_n(x,t).$$

But if $(x,t) \in A_n^*$, $n \geq 0$, $\alpha_n{}^{-1} s^2 \alpha_n(x,t) = h^2(x,t)$ by the construction of α_n. Hence, $h(h(x,t)) = \alpha^{*-1} s \alpha^*(x,t)$ and the claim is established.

Finally, define $\tau \colon S^2 \times (-1,1) \to S^2 \times (0,1)$ by $\tau(x,t) = (x,(t+1)/2)$ and set $\alpha = \tau\alpha^*$. Then $\alpha h \alpha^{-1}(x,t) = (s_1(x),1-t)$, $(x,t) \in S^2 \times (0,1)$.

Case 2. Suppose $S_0 \cap hS_0 = \emptyset$. Let $S_1 = S_0$ and A_0 denote the 3-annulus bounded by S_1 and hS_1 in $S^2 \times (0,1)$. Then A_0 is invariant under h, and by Lemma 3.2 there is a homeomorphism β_0: $A_0 \to S^2 \times [0,1]$ such that $\beta_0 h_0 \beta_0^{-1}(x,t) = (s_1(x), 1-t)$, where $(x,t) \in S^2 \times [0,1]$ and h_0 is the restriction of h to A_0. The construction of the sequence of h^2-invariant 2-spheres, $\{S_n\}_{n=2}^{\infty}$, and the conjugation map, α, such that $\alpha h \alpha^{-1}(x,t) = (s_1(x), 1-t)$ follows exactly the method in case 1b, and hence will be omitted.

Theorem 3.6. Let h: $S^2 \times (0,1) \to S^2 \times (0,1)$ be a Z_4 action on $S^2 \times (0,1)$ with $\text{fix}(h) \neq \emptyset$. Then there is a homeomorphism α: $S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\alpha h \alpha^{-1}(x,t) = (s_2(x), \lambda(t))$, where $x \in S^2$ and $t \in (0,1)$; $\lambda(t) = t$ or $\lambda(t) = 1-t$; and s_2: $S^2 \to S^2$ is the standard period 4 rotation on S^2 leaving the poles fixed.

Proof of Theorem 3.6.

It follows from our preliminary remarks at the beginning of this section that either fix(h) is homeomorphic to S^0 or fix(h) is homeomorphic to two open arcs. Thus, we have two cases to consider.

Case 1. If fix(h) is homeomorphic to two open arcs then $\text{fix}(h) = \text{fix}(h^2)$. Moreover, by Theorem 2.1 there is a homeomorphism α: $S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\alpha h^2 \alpha^{-1}(x,t) = (s_2{}^2(x), t), (x,t) \in S^2 \times (0,1)$. Thus, $(S^2 \times (0,1))/\langle h^2 \rangle$ is homeomorphic to $S^2 \times (0,1)$ and h induces an involution $\bar{h}$: $(S^2 \times (0,1))/\langle h^2 \rangle \to (S^2 \times (0,1))/\langle h^2 \rangle$

such that $p_h h = \bar{h} p_h$, where $p_h\colon S^2 \times (0,1) \to (S^2 \times (0,1)/\langle h^2\rangle$ is the natural quotient map. For simplicity of notation let $s\colon S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_2(x),t)$. Then s also induces an involution $\bar{s}\colon (S^2 \times (0,1))/\langle s^2\rangle \to (S^2 \times (0,1))/\langle s^2\rangle$ such that $p_s s = \bar{s} p_s$, where $p_s\colon S^2 \times (0,1) \to (S^2 \times (0,1))/\langle s^2\rangle$ is the natural quotient map. Since $p_h|_{\mathrm{fix}(h)}\colon \mathrm{fix}(h) \to \mathrm{fix}(\bar{h})$ is a homeomorphism, $\mathrm{fix}(\bar{h})$ is homeomorphic to two open arcs. As in the proof of Theorem 2.1, it follows easily from Theorem 1.1, and Theorem 1.2 that there is a homeomorphism $\bar{\alpha}\colon (S^2 \times (0,1))/\langle h^2\rangle \to (S^2 \times (0,1))/\langle s^2\rangle$ such that $\overline{\alpha h \alpha^{-1}} = \bar{s}$. By using the same method as in the proof of Lemma 3.4, we may lift $\bar{\alpha}$ to a homeomorphism $\alpha\colon [S^2 \times (0,1) - \mathrm{fix}(h)] \to [S^2 \times [0,1] - \mathrm{fix}(s)]$ which may easily be extended to $\mathrm{fix}(h)$ so that $\alpha h \alpha^{-1} = s$ or $\alpha h \alpha^{-1} = s^3$. But since s is conjugate to s^3, h is conjugate to s.

Case 2. Suppose $\mathrm{fix}(h)$ is homeomorphic to S^0. Then there is a collar $B = S^2 \times [0,b]$ of $S^2 \times 0$ in $S^2 \times [0,1]$ such that $S^2 \times (0,b] \cap h(S^2 \times (0,b]) = \emptyset$. Otherwise, there is a sequence $\{(x_n,t_n)\}_{n=1}^{\infty}$ such that (x_n,t_n) and $h(x_n,t_n)$ converge to ∞ in $D = [S^2 \times (0,1) \cup \{-\infty,\infty\}]$, where D is the two point compactification of $S^2 \times (0,1)$. This contradicts the fact that $\mathrm{fix}(h') \cap \{-\infty,\infty\} = \emptyset$ if $\mathrm{fix}(h)$ is homeomorphic to S^0, where $h'\colon D \to D$ is the extension of h to D. By Theorem 2.1, there is a sequence of nontrival 2-spheres, $\{S_n\}_{n=1}^{\infty}$, such that $S_n \subseteq B, h^2 S_n = S_n$ and

$S_{n+1} \subseteq \text{int}(C_n)$, where C_n is the 3-annulus bounded by S_n and $S^2 \times 0$ in $S^2 \times [0,1]$. Also given any collar, B', of $S^2 \times 0$, there is an S_n, $n \geq 1$, such that $S_n \subseteq B'$. Since $S^2 \times (0,b] \cap h(S^2 \times (0,b]) = \emptyset$, $\{hS_n\}_{n=1}^{\infty}$ is a sequence of nontrivial 2-spheres such that $h^2(hS_n) = hS_n$, and $hS_{n+1} \subseteq \text{int}(hC_n)$, where hC_n is the 3-annulus bounded by hS_n and $S^2 \times 1$ in $S^2 \times [0,1]$. The sequence, $\{hS_n\}_{n=1}^{\infty}$, may also be shown to satisfy the additional condition that given any collar, D, of $S^2 \times 1$, there is an $n \geq 1$ such that $hS_n \subseteq D$.

Let A_0 denote the 3-annulus bounded by S_1 and hS_1. Then A_0 is invariant under h and $\text{fix}(h) \subseteq A_0$. Let h_0 denote the restriction of h to A_0. By Lemma 3.3 there is a homeomorphism $\alpha_0 \colon A_0 \to S^2 \times [0,1]$ such that $\alpha_0 h_0 \alpha_0^{-1}(x,t) = (s_2(x), 1-t)$, where $(x,t) \in S^2 \times [0,1]$. Define $\delta \colon S^2 \times [0,1] \to S^2 \times [-1/2,1/2]$ by $\delta(x,t) = (x, t-(1/2))$, and let $\alpha_0^* = \delta\alpha_0$. Then $\alpha_0^* h_0 \alpha_0^{*-1}(x,t) = (s_2(x), -t)$, where $(x,t) \in S^2 \times [-1/2,1/2]$. Let A_n denote the 3-annulus bounded by S_n and S_{n+1}, $n \geq 1$. By the construction of the sequence $\{S_n\}_{n=1}^{\infty}$, $S^2 \times (0,1) = (\cup_{n=0}^{\infty} A_n) \cup (\cup_{n=0}^{\infty} hA_n)$, and each A_n is invariant under h^2. The piecewise definition of the conjugation map $\alpha \colon S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\alpha h \alpha^{-1}(x,t) = (s_2(x), 1-t)$ follows exactly the methods used in Theorem 3.5, case 1b, and is thus omitted.

It follows from Theorem 3.5 and 3.6 and our preliminary remarks that all Z_4 actions on $S^2 \times (0,1)$ are classified. Hence, we have the following corollary.

Corollary. $S^2 \times (0,1)$ admits exactly four nonequivalent Z_4 actions up to conjugation.

CHAPTER FOUR
2^n ACTIONS ON $S^2 \times (0,1)$ AND R^3

In this chapter we classify all cyclic 2^n actions on $S^2 \times (0,1)$ and R^3. In the first section, the results obtained in Chapter Three for Z_4 actions on $S^2 \times (0,1)$ are extended to 2^n actions on $S^2 \times (0,1)$ for $n \geq 2$ by a simple application of the Lifting Theorem. In the second section, we show that R^3 admits exactly two nonequivalent Z_{2^n} actions up to weak conjugation. The method will be to first classify Z_4 actions on R^3 up to conjugation, and extend this classification to Z_{2^n} actions, $n>2$, via the Lifting Theorem (Theorem 5.1, [13]).

2^n Actions on $S^2 \times (0,1)$

We first describe the standard 2^n actions on $S^2 \times (0,1)$. View S^2 as a subset of R^3 in the following manner: $S^2 = \{(x_1,x_2,x_3) \in R^3 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}$. Now let $q = 2^n$, $n \geq 2$, where n is fixed. Define $s_1\colon S^2 \to S^2$ by $s_1(x_1,x_2,x_3) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2, x_3)$ and $s_2\colon S^2 \to S^2$ by $s_2(x_1,x_2,x_3) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2, -x_3)$, where $\alpha = \cos(2\pi/q)$ and $B = \sin(2\pi/q)$. The homeomorphisms s_1 and s_2 may also be described in terms of cylindrical coordinates: $s_1(r,\theta,x_3) = (r,\theta + 2\pi/q, x_3)$ and $s_2(r,\theta,x_3) = (r,\theta + 2\pi/q, -x_3)$, where $(r,\theta,x_3) \in S^2$. Thus, s_1 is the standard period 2^n rotation

on S^2 leaving the poles fixed, and s_2 is the standard period 2^n rotation leaving the poles fixed followed by a reflection through the equator. Let $\lambda_1: (0,1) \to (0,1)$ be the identity map and $\lambda_2: (0,1) \to (0,1)$ be defined by $\lambda_2(t) = 1-t$, $t \in (0,1)$. Then $\langle s_i \times \lambda_j \rangle$, $i = 1,2$, $j = 1,2$, is the group of standard 2^n actions on $S^2 \times (0,1)$. We will show that given a cyclic action, $h: S^2 \times (0,1) \to S^2 \times (0,1)$, of period 2^n, $n \geq 2$, h is weakly conjugate to $s_i \times \lambda_j$, $i = 1$ or $i = 2$, $j = 1$ or $j = 2$.

Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a cyclic action on $S^2 \times (0,1)$, $n \geq 2$. If $n = 2$, it follows from Theorem 2.1 that $\mathrm{fix}(h^2)$ is homeomorphic to two open arcs. In particular, there are no free Z_4 actions on $S^2 \times (0,1)$. Hence, it follows easily that there are no free cyclic 2^n actions for $n>2$. The following theorem, although not needed in the sequel, provides a more general result in this direction.

Theorem 4.1. For $n>2$, there are no free Z_n actions on $S^2 \times (0,1)$.

Proof of Theorem 4.1.

It follows from our preliminary remarks that there are no free Z_{2^n} actions on $S^2 \times (0,1)$ for $n \geq 2$. Now suppose there is a free Z_q action, $h: S^2 \times (0,1) \to S^2 \times (0,1)$, where q is a prime number greater than 2. Let $M = (S^2 \times (0,1))/\langle h \rangle$ and let $p: S^2 \times (0,1) \to M$ denote the natural quotient map. Since h is free, M is a

3-manifold and p is a covering map. Now $\pi_2(S^2 \times (0,1)) = Z$ and $p_*: \pi_2(S^2 \times (0,1)) \to M$ is an isomorphism. Hence, $\pi_2(M) \neq 0$ and by the Generalized Sphere Theorem [4], M contains a 2-sphere or two-sided projective plane which is nontrivial in M. We denote this surface by F. Since p is a covering map and $S^2 \times (0,1)$ contains no projective plane, each arc component of $p^{-1}(F)$ is homeomorphic to S^2. Let F^* be an arc component of $p^{-1}(F)$, and $p_* = p|_{F^*}$. If F is homeomorphic to P^2, where P^2 is the two-dimensional projective plane, then p^* is a 2-1 cover, and there is a deck transformation $g: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $g(F^*) = F^*$. Thus g is an involution and $g \in \langle h\rangle$, contradicting the fact that every element of the group $\langle h\rangle$ is of prime order q. Hence, F is homeomorphic to S^2 and p^* is a 1-1. Now $p^{-1}F = \bigcup_{i=1}^{4} F_i^*$, where each F_i^* is a nontrivial 2-sphere in $S^2 \times (0,1)$. We may assume that the arc components, $\{F_i^*\}_{i=1}^{q}$, have been labelled so that the 3-annulus, A_i, bounded by F_i^* and $S^2 \times 0$ in $S^2 \times [0,1]$ has the property that $A_i \cap (p^{-1}F - (\bigcup_{j\leq i} F_j^*)) = \emptyset$, $i,j = 1,2,3,4$. Now, there is a deck transformation, $h': S^2 \times (0,1) \to S^2 \times (0,1)$ such that $h'F_1^* = F_2^*$. By our choice of labelling, F_1^* separates $S^2 \times (0,1)$ into two open 3-annuli, B_1 and B_2, such that $B_1 \cap p^{-1}F = \emptyset$ or $B_2 \cap p^{-1}F = \emptyset$. But $h'F_1^* = F_2^*$ separates $S^2 \times (0,1)$ into $h'B_1$ and $h'B_2$ with $h'B_i \cap p^{-1}F \neq \emptyset$, $i = 1,2$. This is a contradiction since the set $p^{-1}F$ is invariant under h'. Thus, there are no free Z_q actions on $S^2 \times (0,1)$ where q is a prime.

Now suppose there is a free Z_n action, $h: S^2 \times (0,1) \to S^2 \times (0,1)$ where $n>2$. We factor n into $n = 2^m p_1 p_2 \cdots p_k$, where the p_i are primes distinct from 2 and $m \geq 0$. If $n = 2^m$, $m \geq 2$, then h is a free Z_{2^m} action which is a contradiction by our preliminary remarks. If $n \neq 2^m$, then h^k is a free action of prime period $p_1 > 2$, where $k = 2^m p_2 p_3 \cdots p_k$, again a contradiction. Hence, there are no free Z_n actions on $S^2 \times (0,1)$ for $n>2$.

Now let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a cyclic 2^n action on $S^2 \times (0,1)$, $n \geq 2$, and let

$$D = [S^2 \times (0,1) \cup \{-\infty,\infty\}]$$

denote the two point compactification of $S^2 \times (0,1)$. Then D is homeomorphic to S^3 and h has an extension $h': D \to D$. Moreover, $(h')^{k_1}$ is an involution with $\{-\infty,\infty\} \subseteq \operatorname{fix}((h')^{k_1})$, where $k_1 = 2^{n-1}$. By Smith's Theorem [18], $\operatorname{fix}((h')^{k_1})$ is homeomorphic to S^1, and thus $\operatorname{fix}(h^{k_1})$ is homeomorphic to two open arcs. If $n>2$, $(h')^{k_2}$ is an orientation preserving homeomorphism with $\{-\infty,\infty\} \subseteq \operatorname{fix}((h')^{k_2})$, where $k_2 = 2^{n-2}$. Thus $\operatorname{fix}((h')^{k_2})$ is homeomorphic to S^1 and $\operatorname{fix}(h^{k_2})$ is homeomorphic to open arcs. But $\operatorname{fix}(h^{k_2}) \subseteq \operatorname{fix}(h^{k_1})$ and $\operatorname{fix}(h^{k_2})$ is contained as a closed subset of $\operatorname{fix}(h^{k_1})$, and it follows that $\operatorname{fix}(h^{k_2}) = \operatorname{fix}(h^{k_1})$. Applying this reasoning $n-1$ times we obtain $\operatorname{fix}(h^2) = \operatorname{fix}(h^4) = \ldots = \operatorname{fix}(h^{k_1})$, and $\operatorname{fix}(h^{k_1})$ is homeomorphic to two open arcs. The following lemma classifies h^2 up to weak conjugation.

Lemma 4.2. Let $g: S^2 \times (0,1) \to S^2 \times (0,1)$ be a cyclic 2^n action on $S^2 \times (0,1)$ such that fix(g) is homeomorphic to two open arcs and $\text{fix}(g) = \text{fix}(g^k)$, $1 \le k \le 2^n - 1$. Then there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle \alpha g \alpha^{-1} \rangle = \langle s_1 \times \lambda_1 \rangle$ where $s_1: S^2 \to S^2$ and $\lambda_2: (0,1) \to (0,1)$ are as defined above.

Proof of Lemma 4.2.

If $n = 1$, g is an involution, and the conclusion easily follows from Theorem 1.1 and Theorem 1.2. Assume the lemma is true for 2^{n-1} actions, and let $g: S^2 \times (0,1) \to S^2 \times (0,1)$ be a cyclic 2^n action with $\text{fix}(g) = \text{fix}(g^k)$, $1 \le k \le 2^n - 1$, where fix(g) is homeomorphic to two open arcs. By our induction hypothesis, $(S^2 \times (0,1))/\langle g^2 \rangle$ is homeomorphic to $S^2 \times (0,1)$, and g induces a homeomorphism $g^*: (S^2 \times (0,1))/\langle g^2 \rangle \to (S^2 \times (0,1))/\langle g^2 \rangle$ which makes the following diagram commute:

$$\begin{array}{ccc} S^2 \times (0,1) & \xrightarrow{g} & S^2 \times (0,1) \\ p \downarrow & & p \downarrow \\ (S^2 \times (0,1))/\langle g^2 \rangle & \xrightarrow{g^*} & (S^2 \times (0,1))/\langle g^2 \rangle, \end{array}$$

where $p: S^2 \times (0,1) \to (S^2 \times (0,1))/\langle g^2 \rangle$ is the natural quotient map. Let $s: S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_1(x), \lambda_1(t))$. Then s also induces a homeomorphism $s^*: (S^2 \times (0,1)/\langle s^2 \rangle \to (s^2 \times (0,1))/\langle s^2 \rangle$ such that $p_1 s = s^* p_1$, where $p_1: S^2 \times (0,1) \to (S^2 \times (0,1))/\langle s^2 \rangle$ is the natural quotient map. By case $n = 1$, there is a

homeomorphism α': $(S^2 \times (0,1))/\langle g^2\rangle \to (S^2 \times (0,1))/\langle s^2\rangle$ such that $\alpha' g^* \alpha'^{-1} = s^*$. Since $\alpha'_* p_*(\pi_1(S^2 \times (0,1) - \mathrm{fix}(g))) = p_{1*}(\pi_1(S^2 \times (0,1) - \mathrm{fix}(s)))$, by the Lifting Theorem, there is a homeomorphism α: $[S^2 \times (0,1) - \mathrm{fix}(h)] \to [S^2 \times (0,1) - \mathrm{fix}(s)]$, which can be easily extended to $\mathrm{fix}(h)$, so that $\langle \alpha h \alpha^{-1}\rangle = \langle s\rangle$. Hence, the lemma is established.

The following two theorems classify all Z_2n actions on $S^2 \times (0,1)$, $n>2$.

Theorem 4.3. Let h: $S^2 \times (0,1) \to S^2 \times (0,1)$ be a semifree Z_2n action on $S^2 \times (0,1)$, $n \geq 2$. Then there is a homeomorphism α: $S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle \alpha h \alpha^{-1}\rangle = \langle s_2 \times \lambda_1\rangle$ or $\langle \alpha h \alpha^{-1}\rangle = \langle s_2 \times \lambda_2\rangle$.

Proof of Theorem 4.3.

Let h: $S^2 \times (0,1) \to S^2 \times (0,1)$ be a semifree Z_2n action on $S^2 \times (0,1)$. If $n = 2$, the conclusion follows from Theorem 3.5, and hence we assume $n>2$. By our preliminary remarks, it follows that $\mathrm{fix}(h^2) = \mathrm{fix}(h^4) = \ldots = \mathrm{fix}(h^{k_1})$, where $k_1 = 2^{n-1}$, and $\mathrm{fix}(h^{k_1})$ is homeomorphic to two open arcs. By Lemma 4.2, $(S^2 \times (0,1))/\langle h^4\rangle$ is homeomorphic to two open arcs. There are two cases to consider.

Case 1. Consider $S^2 \times (0,1)$ as a subset of $S^2 \times [0,1]$ and suppose that for every collar $C = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$, $S^2 \times (0,c] \cap h(S^2 \times (0,c]) \neq \emptyset$. Let s: $S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_2(x), \lambda_1(t))$, where s_2: $S^2 \to S^2$ and λ_1: $(0,1) \to (0,1)$

are as defined above. Let $p_s\colon S^2 \times (0,1) \to (S^2 \times (0,1))/\langle s^4\rangle$, and $p_h\colon S^2 \times (0,1) \to (S^2 \times (0,1))/\langle h^4\rangle$ be the natural quotient maps. Now, h induces a homeomorphism $\bar{h}\colon (S^2 \times (0,1))/\langle h^4\rangle \to (S^2 \times (0,1))/\langle h^4\rangle$ and s induces the standard homeomorphism $\bar{s}\colon (S^2 \times (0,1))/\langle s^4\rangle \to (S^2 \times (0,1))/\langle s^4\rangle$ which make the following diagrams commute:

$$\begin{array}{ccc} S^2 \times (0,1) & \xrightarrow{h} & S^2 \times (0,1) \\ p_h \downarrow & & p_h \downarrow \\ (S^2 \times (0,1))/\langle h^4\rangle & \xrightarrow{\bar{h}} & (S^2 \times (0,1))/\langle h^4\rangle \end{array}$$

$$\begin{array}{ccc} S^2 \times (0,1) & \xrightarrow{s} & S^2 \times (0,1) \\ p_s \downarrow & & p_s \downarrow \\ (S^2 \times (0,1))/\langle s^4\rangle & \xrightarrow{\bar{s}} & (S^2 \times (0,1))/\langle s^4\rangle . \end{array}$$

By Lemma 4.3, $(S^2 \times (0,1))/\langle h^4\rangle$ is homeomorphic to $S^2 \times (0,1)$ and it can be easily shown, using the above diagram, that given any collar $c = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$, $p_h(S^2 \times (0,c]) \cap \bar{h}(p_h(S^2 \times (0,c])) \neq \emptyset$. Hence, by Theorem 3.5, there is a homeomorphism $\bar{\alpha}\colon (S^2 \times (0,1))/\langle h^4\rangle \to (S^2 \times (0,1))/\langle s^4\rangle$ such that $\overline{\alpha h \alpha}^{-1} = \bar{s}$. Since

$$\bar{\alpha}_* p_h{}^*(\pi_1(S^2 \times (0,1) - \mathrm{fix}(h^2))) = p_{s*}(\pi_1(S^2 \times (0,1) - \mathrm{fix}(s^2))),$$

by the Lifting Theorem, there is a homeomorphism $\alpha\colon [S^2 \times (0,1) - \mathrm{fix}(h^2)] \to [S^2 \times (0,1) - \mathrm{fix}(s^2]$ such that $\langle \alpha h \alpha^{-1}\rangle = \langle s|_{[S^2 \times (0,1) - \mathrm{fix}(s^2)]}$. Since

$p_h\big|_{\text{fix}(h^2)}: \text{fix}(h^2) \to \text{fix}(\bar{h}^2)$, $p_s\big|_{\text{fix}(s^2)}: \text{fix}(s^2) \to \text{fix}(\bar{s}^2)$, and $\bar{\alpha}\big|_{\text{fix}(\bar{h}^2)}: \text{fix}(\bar{h}^2) \to \text{fix}(\bar{s}^2)$ are homeomorphisms, we may extend α: $S^2 \times (0,1)$ by defining $\alpha(x,t) = p_s^{-1}\bar{\alpha}p_h(x,t)$ for $(x,t) \in \text{fix}(h^2)$. Thus, we have a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle\alpha h\alpha^{-1}\rangle = \langle s_2 \times \lambda_1\rangle$.

Case 2. Suppose there is a collar $C = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$ such that $S^2 \times (0,c] \cap h(S^2 \times (0,c]) = \emptyset$. Let $s: S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_2(x),\lambda_2(t))$. By a completely analogous argument to that in case 1, there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle\alpha h\alpha^{-1}\rangle = \langle s\rangle$. This completes the proof of Theorem 4.3.

Theorem 4.4. Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a Z_{2^n} action on $S^2 \times (0,1)$ with $\text{fix}(h) \neq \emptyset$, $n \geq 2$. Then there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle\alpha h\alpha^{-1}\rangle = \langle s_1 \times \lambda_1\rangle$ or $\langle\alpha h\alpha^{-1}\rangle = \langle s_1 \times \lambda_2\rangle$.

Proof of Theorem 4.4.

Let $h: S^2 \times (0,1) \to S^2 \times (0,1)$ be a cyclic 2^n action on $S^2 \times (0,1)$ with $\text{fix}(h) \neq \emptyset$, $n \geq 2$. If $n = 2$, the conclusion follows from Theorem 3.6, and hence we assume $n>2$. Again, it follows from our preliminary remarks that $\text{fix}(h^2) = \text{fix}(h^4) = \ldots = \text{fix}(h^{k_1})$, where $k_1 = 2^{n-1}$, and $\text{fix}(h^{k_1})$ is homeomorphic to two open arcs. By Lemma 4.2, $(S^2 \times (0,1))/\langle h^4\rangle$ is homeomorphic to $S^2 \times (0,1)$. There are two cases to consider.

Case 1. Suppose there is a collar $C = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,c]$ such that $S^2 \times (0,c] \cap h(S^2 \times (0,c]) = \emptyset$. Let $s: S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_1(x),\lambda_2(t))$, and $p_h: S^2 \times (0,1) \to (S^2 \times (0,1))/\langle h^4\rangle$ and $p_s: S^2 \times (0,1) \to (S^2 \times (0,1))/\langle s^4\rangle$ denote the natural quotient maps. Now, h induces a homeomorphism $\bar{h}: (S^2 \times (0,1))/\langle h^4\rangle \to (S^2 \times (0,1))/\langle h^4\rangle$ such that $\bar{h}p_h = p_h h$, with $\text{fix}(h) \neq \emptyset$. Similarly, s induces the standard homeomorphism $\bar{s}: (S^2 \times (0,1))/\langle s^4\rangle \to (S^2 \times (0,1))/\langle s^4\rangle$ such that $\bar{s}p_s = p_s s$. By Theorem 3.6, there is a homeomorphism $\bar{\alpha}: (S^2 \times (0,1))/\langle h^4\rangle \to (S^2 \times (0,1))/\langle s^4\rangle$ such that $\overline{\alpha h \alpha^{-1}} = \bar{s}$. Since $p_{h*}\bar{\alpha}_*(\pi_1(S^2 \times (0,1) - \text{fix}(h^2))) = p_{s*}(\pi_1(S^2 \times (0,1) - \text{fix}(s^2)))$, by the Lifting Theorem, there is a homeomorphism $\alpha: [S^2 \times (0,1) - \text{fix}(h^2)] \to [S^2 \times (0,1) - \text{fix}(s^2)]$, which can be easily extended to $\text{fix}(h^2)$ so that $\langle \alpha h \alpha^{-1}\rangle = \langle s\rangle$.

Case 2. Suppose for every collar $c = S^2 \times [0,c]$ of $S^2 \times 0$ in $S^2 \times [0,1]$, $S^2 \times (0,c] \cap h(S^2 \times (0,c]) \neq \emptyset$. Let $s: S^2 \times (0,1) \to S^2 \times (0,1)$ be defined by $s(x,t) = (s_1(x),\lambda_1(t))$. By using an argument completely analogous to that used in case 1, or appealing to Lemma 4.2 directly, there is a homeomorphism $\alpha: S^2 \times (0,1) \to S^2 \times (0,1)$ such that $\langle \alpha h \alpha^{-1}\rangle = \langle s\rangle$. This completes the proof of Theorem 4.4.

It follows easily from Theorem 1.1 and Theorem 1.2 that there are seven nonequivalent involutions on $S^2 \times (0,1)$, and it follows from Lemma 4.1, Theorem 4.3, and Theorem 4.4

that there are four nonequivalent Z_2n actions on $S^2 \times (0,1)$ for $n \geq 2$. We thus have a complete classification of all Z_2n actions on $S^2 \times (0,1)$, $n \geq 1$.

Cyclic 2^n Actions on R^3

We first describe the standard actions of period 2^n on R^3, $n \geq 2$. Let $(r,\theta,z) \in R^3$ be expressed in cylindrical coordinates and $q = 2^n$, where n is fixed. Define $s_1\colon R^3 \to R^3$ by $s_1(r,\theta,z) = (r,\theta+2\pi/q,z)$ and $s_2\colon R^3 \to R^3$ by $s_2(r,\theta,z) = (r,\theta+2\pi/q,-z)$. Thus s_1 is the standard period 2^n rotation about the z-axis, and s_2 is the standard period 2^n rotation about the z-axis followed by a reflection through the xy-plane. An alternate definition of s_1 and s_2 in terms of rectangular coordinates is the following:

$$s_1(x_1,x_2,x_3) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2, x_3) \text{ and}$$

$$s_2(x_1,x_2,x_3) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2, -x_3),$$

$(x_1,x_2,x_3) \in R^3$, where $\alpha = \cos(2\pi/q)$ and $\beta = \sin(2\pi/q)$. Then $\langle s_1,s_2 \rangle$ is the group of standard 2^n actions on R^3. We will show that given a cyclic action, $h\colon R^3 \to R^3$, of period 2^n, $n \geq 2$, h is weakly conjugate to s_1 or s_2.

Let $h\colon R^3 \to R^3$ be a cyclic action of period 2^n on R^3, $n \geq 2$. By taking the one point compactification of R^3, we obtain an extension, $h'\colon S^3 \to S^3$, of h with $\text{fix}(h') \neq \emptyset$. Since $(h')^2\colon S^3 \to S^3$ is an orientation preserving homeomorphism of period 2^{n-1}, it follows from Smith's Theorem

[18], that $\mathrm{fix}((h')^2)$ is homeomorphic to S^1. Since $\mathrm{fix}(h') \neq \emptyset$, $\mathrm{fix}(h')$ is homeomorphic to S^i, $i = 0,1,2$ (see [18]). But $\mathrm{fix}(h') \subseteq \mathrm{fix}((h')^2)$, and thus either $\mathrm{fix}(h')$ is homeomorphic to S^0 or S^1. It follows that either $\mathrm{fix}(h)$ is homeomorphic to a single point or an open arc. The next two lemmas establish our results for the special case where $n = 2$.

Lemma 4.5. Let $h: R^3 \to R^3$ be a Z_4 action on R^3 with $\mathrm{fix}(h^k)$ homeomorphic to an open arc, $1 \le k < 4$. Then there is a homeomorphism $\alpha: R^3 \to R^3$ such that $\alpha h \alpha^{-1} = s_1$, where $s_1(x_1,x_2,x_3) = (-x_2,x_1,x_3)$.

Proof of Lemma 4.5.

By Theorem 1.1, h^2 is conjugate to the standard orientation preserving involution on R^3, and thus $R^3/\langle h^2\rangle$ is homeomorphic to R^3. Let $p_h: R^3 \to R^3/\langle h^2\rangle$ and $p_s: R^3 \to R^3/\langle s_1{}^2\rangle$ denote the natural quotient maps. Now h induces an involution $\bar{h}: R^3/\langle h^2\rangle \to R^3/\langle h^2\rangle$ such that $p_h h = \bar{h} p_h$ with $\mathrm{fix}(\bar{h})$ homeomorphic to an open arc, and s_1 induces the standard orientation preserving involution $\bar{s}_1: R^3/\langle s_1{}^2\rangle \to R^3/\langle s_1{}^2\rangle$ such that $ps_1 = \bar{s}_1 p$. Again applying Theorem 1.1, there is a homeomorphism $\bar{\alpha}: R^3/\langle h^2\rangle \to R^3/\langle s_1{}^2\rangle$ such that $\bar{\alpha}\bar{h}\bar{\alpha}^{-1} = \bar{s}_1$. Now $\bar{\alpha}_* p_{h*}(\pi_1(R^3 - \mathrm{fix}(h^2))) = p_{s*}(\pi_1(R^3 - \mathrm{fix}(s_1{}^2)))$ and by the Lifting Theorem, there is a homeomorphism $\alpha': [R^3 - \mathrm{fix}(h^2)] \to [R^3 - \mathrm{fix}(s_1{}^2)]$ which can be easily extended to a homeomorphism $\alpha: R^3 \to R^3$ such that

$\alpha h\alpha^{-1} = s_1$ or $\alpha h\alpha^{-1} = s_1{}^3$. But s_1 and $s_1{}^3$ are conjugate, and thus the lemma is established.

Lemma 4.6. Let $h: R^3 \to R^3$ be a Z_4 action on R^3 with fix(h) consisting of a single point. Then h is conjugate to s_2 where $s_2(x_1,x_2,x_3) = (-x_2,x_1,-x_3)$.

Proof of Lemma 4.6.

By our preliminary remarks $\text{fix}(h^2)$ is homeomorphic to an open arc. Now let N be a regular neighborhood of fix(h) with $\partial N = S_0$. Then $R^3 - N$ is homeomorphic to $S^2 \times (0,1)$ and $h|_{Cl(R^3-N)}: Cl(R^3 - N) \to Cl(R^3 - N)$ is a Z_4 action on $Cl(R^3 - N)$ with $\text{fix}(h|_{Cl(R^3-N)}) = \emptyset$, and $\text{fix}(h^2|_{Cl(R^3-N)})$ homeomorphic to two half-open arcs. Using the methods of Lemma 2.4, we can construct a 2-sphere $S_1 \subseteq R^3 - N$ such that S_1 does not bound a 3-cell in $R^3 - N$ and either $hS_1 = S_1$ or $hS_1 \cap S_1 = \emptyset$ Since N is invariant under h, $hS_1 \cap S_1 = \emptyset$ cannot occur, and thus $hS_1 = S_1$; S_1 divides R^3 into an open 3-ball, B_2, and an open 3-annulus, A_2, such that $hA_2 = A_2$. Again, there is an h-invariant 2-sphere $S_2 \subseteq A_2$ such that S_2 does not bound a 3-cell in A_2. Continuing in this manner we construct a sequence, $\{S_n\}_{n=1}^{\infty}$, of h-invariant 2-spheres such that $S_n \subseteq A_n$, $n \geq 1$, where $A_1 = Cl(R^3 - N)$, and S_n does not bound a 3-cell in A_n. Now let C_n denote the 3-annulus bounded by S_{n-1} and S_n, $n \geq 1$. We may assume that the sequence $\{S_n\}_{n=1}^{\infty}$ is chosen so that $N \cup (\cup_{n=1}^{\infty} C_n) = R^3$. It follows easily from Lemma 3.2 that there is a homeomorphism $\alpha_n': C_n \to D_n$, $n \geq 1$,

such that $\alpha_n' h_n \alpha_n'^{-1} = s_2|_{D_n}$, where h_n is the restriction of h to C_n and $D_n = \{(x_1,x_2,x_3) \in R^3 \mid n \leq (x_1)^2 + (x_2)^2 + (x_3)^2 \leq (n+1)\}$. Moreover, by using methods similar to those in the proof of Lemma 3.2, case 2, each α_n' may be altered to a homeomorphism $\alpha_n\colon C_n \to D_n$ such that $\alpha_n|_{S_n} = \alpha_{n+1}|_{S_n}$, $n \geq 1$. Define $\alpha\colon Cl(R^3 - N) \to Cl(R^3 - B_1)$ by $\alpha(x_1,x_2,x_3) = \alpha_n(x_1,x_2,x_3)$ for $(x_1,x_2,x_3) \in C_n$, where B_1 denotes the standard unit 3-ball in R^3. Finally, α may be extended to N by coning over the boundary of N to obtain a homeomorphism $\alpha\colon R^3 \to R^3$ such that $\alpha h \alpha^{-1} = s_2$.

Combining Lemmas 4.5 and 4.6, we have the following corollary.

Corollary. R^3 admits exactly two nonequivalent Z_4 actions up to conjugation.

The following theorem classifies all Z_{2^n} actions on R^3 for $n \geq 2$ up to weak conjugation.

Theorem 4.7. Let $h\colon R^3 \to R^3$ be a cyclic action of period 2^n, $n \geq 2$. Then there is a homeomorphism $\alpha\colon R^3 \to R^3$ such that $\langle \alpha h \alpha^{-1} \rangle = \langle s_1 \rangle$ or $\langle \alpha h \alpha^{-1} \rangle = \langle s_2 \rangle$, where s_1 and s_2 are as defined above.

Proof of Theorem 4.7.

By our preliminary remarks, fix(h) is either a single point or an open arc. We thus have two cases to consider.

Case 1. Suppose fix(h) is homeomorphic to an open arc. Then fix(h^k) is homeomorphic to an open arc for $1 \leq k < n$. The

existence of the conjugation map, α, such that $\langle \alpha h \alpha^{-1} \rangle = \langle s_1 \rangle$ follows easily by induction using a method similar to that used in the proof of Lemma 4.5 and is thus omitted.

Case 2. Suppose fix(h) is a single point. If $n = 2$, the conclusion follows from Lemma 4.6, and hence we assume that $h\colon R^3 \to R^3$ is a cyclic action of period 2^n, $n>2$. Now h^2 is an orientation preserving homeomorphism on R^3 and thus $\mathrm{fix}(h^2)$ is homeomorphic to an open arc. It can also be shown that $\mathrm{fix}(h^2) = \mathrm{fix}(h^4) = \ldots = \mathrm{fix}(h^{k_1})$, where $k_1 = 2^{n-1}$. Thus, by case 1, h^4 is standard and $R^3/\langle h^4 \rangle$ is homeomorphic to R^3. Let $s_2\colon R^3 \to R^3$ be as defined above, and let $p_h\colon R^3 \to R^3/\langle h^4 \rangle$ and $p_s\colon R^3 \to R^3/\langle {s_2}^4 \rangle$ denote the natural quotient maps. Both $\overline{h}$ and $\overline{s}_2$ induce homeomorphisms h and s_2 such that $p_h h = \overline{h} p_h$ and $p_s s_2 = \overline{s}_2 p_s$. By Lemma 4.6, there is a homeomorphism $\alpha\colon R^3/\langle h^4 \rangle \to R^3/\langle {s_2}^4 \rangle$ such that $\overline{\alpha}\overline{h}\overline{\alpha}^{-1} = \overline{s}_2$. Since $\overline{\alpha}_* p_{h*}(\pi_1(R^3 - \mathrm{fix}(h^2))) = p_{s*}(\pi_1(R^3 - \mathrm{fix}({s_2}^2)))$, by the Lifting Theorem there is a homeomorphism $\alpha\colon [R^3 - \mathrm{fix}(h^2)] \to [R^3 - \mathrm{fix}({s_2}^2)]$ such that $\langle \alpha h \alpha^{-1} \rangle = \langle s_2|_{R^3 - \mathrm{fix}(s_2{}^2)} \rangle$. Since p_h and p_s are one-to-one on $\mathrm{fix}(h^2)$ and $\mathrm{fix}({s_2}^2)$ respectively, α can be easily extended to $\mathrm{fix}(h^2)$ by defining $\alpha(x_1,x_2,x_3) = {p_s}^{-1}\overline{\alpha} p_n(x_1,x_2,x_3)$ for $(x_1,x_2,x_3) \in \mathrm{fix}(h^2)$. Thus, there is a homeomorphism $\alpha\colon R^3 \to R^3$ such that $\langle \alpha h \alpha^{-1} \rangle = \langle s_2 \rangle$. The completes the proof of Theorem 1.

In [12], Kwun and Tollefson established the uniqueness of orientation preserving PL involutions of Euclidean

3-space, and it follows easily from a more general result by the same authors (see [11]) that there are two orientation reversing PL involutions up to conjugation. Thus, R^3 admits exactly three nonequivalent PL involutions up to conjugation. This combined with Theorem 4.7 gives a complete classification of 2^n actions on R^3, $n \geq 1$, up to weak conjugation.

REFERENCES

1. J. W. Alexander, "A Proof and Extension of the Jordan-Brouwer Separation Theorem," Trans. Amer. Math. Soc., Volume 23 (1922), 333-349.

2. R. H. Bing, "An Alternative Proof That 3-Manifods Can Be Triangulated," Ann. of Math., Volume 69 (1959), 37-65.

3. E. M. Brown, "Unknotting in $M^2 \times I$," Trans. Amer. Math. Soc., Volume 123 (1966), 480-505.

4. D. B. A. Epstein, "Projective Planes in 3-Manifolds," Proc. London Math. Soc. (3), Volume 11 (1961), 469-484.

5. J. Hempel, 3-Manifolds, Ann. of Math. Studies, Princeton Univ. Press, Princeton, 1976.

6. D. W. Henderson, "Relative General Position," Pacific J. Math., Volume 18 (1966), 513-523.

7. J. F. P. Hudson, Piecewise Linear Topology, W. A. Benjamin, New York, 1969.

8. P. K. Kim, "Cyclic Actions on Lens Spaces," Trans. Amer. Math. Soc., Volume 237 (1978), 121-144.

9. __________, "Periodic Homeomorphisms of the 3-Sphere," Michigan Math. J., Volume 21 (1974), 1-5.

10. P. K. Kim and J. L. Tollefson, "PL Involutions of Fibered 3-Manifolds," Trans. Amer. Math. Soc., Volume 232 (1977), 221-237.

11. K. W. Kwun and J. L. Tollefson, "Extending a PL Involution of the Interior of a Compact Manifold," Amer. J. Math., Volume 99 (1977), 995-1001.

12. __________, "Uniqueness of Orientation Preserving PL Involutions of 3-Space," Bull. Amer. Math. Soc., Volume 79 (1973), 761-762.

13. W. S. Massey, Algebraic Topology, and Introduction, Springer-Verlag, New York, 1977.

14. E. E. Moise, "Affine Structures in 3-Manifolds V. The Triangulation Theorem and Hauptvermutung," Ann. of Math., Volume 55 (1952), 96-114.

15. P. M. Rice, "Free Actions of Z_4 on S^3," Duke Math. J. Volume 36 (1969), 749-751.

16. G. X. Ritter, "Free Z_8 Actions on S^3," Trans. Amer. Math. Soc., Volume 181 (1973), 195-212.

17. __________ and B. E. Clark, "Periodic Homeomorphisms on Noncompact 3-Manifolds," London Mathematical Society Lecture Note Series, Volume 26 (1976), 154-168.

18. P. A. Smith, "Transformations of Finite Period II," Ann. of Math., Volume 40 (1939), 690-711.

19. Y. Tao, "On Fixed Point Free Involutions of $S^1 \times S^2$," Osaka J. Math., Volume 14 (1962), 145-152

20. J. L. Tollefson, "The Compact 3-Manifolds Covered by $S^2 \times R^1$," Proc. Amer. Math. Soc., Volume 43 (1974), 461-462.

21. F. Waldhausen, "Über Involutionen der 3-Sphäre," Topology 8 (1969), 81-92.

BIOGRAPHICAL SKETCH

David G. Winslow was born in Tampa, Florida, on October 28, 1948. In 1970, he received a Bachelor of Arts degree in mathematics from the University of South Florida. Following a three-year tour in the United States Air Force, he entered the University of Florida in January, 1974, and received a Master of Science degree in mathematics in 1976. Since 1975, he has studied and taught mathematics at the University of Florida and is a member of the American Mathematical Society and Phi Kappa Phi. He is married to Alicia Browner Winslow.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Gerhard Ritter

Gerhard X. Ritter, Chairman
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark L. Teply

M. L. Teply
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

JK Brooks

J. K. Brooks
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

D. C. Wilson

D. C. Wilson
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J. W. Gregory
Associate Professor of Education

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 1979

Dean, Graduate School